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PLANE GEOMETRY

BY

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BENJ. H. SANBORN & CO.
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1916

STONE–MILLIS MATHEMATICS SERIES

ELEMENTARY ARITHMETICS:
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HIGHER ALGEBRA
PLANE GEOMETRY
SOLID GEOMETRY
PLANE AND SOLID GEOMETRY

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Mathematics.
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PREFACE

THE STONE-MILLIS GEOMETRY, published in 1910, was a pioneer in its field, being the first of the series of American textbooks on geometry which of recent years have attempted in various ways to modernize the teaching of the subject. It marked, in fact, a wide departure from the traditional Greek geometry, after which textbooks for secondary schools had for generations been patterned. From the start, the text met with remarkable success. The educational ideals which it embodied are now national ideals, and are summarized in the REPORT OF THE NATIONAL COMMITTEE OF FIFTEEN on the teaching of geometry.

The present volume, by the same authors, is a thorough revision of that text. It has been prepared in the attempt to produce a text which shall preserve the distinctive features of the older book, but which, if possible, shall be more simple, practical, and teachable.

The main features of the older book, which have been preserved in the present volume, are:

(1) A concrete approach to the subject, developing a body of experience and imagery as a basis of formal geometry, the latter not being introduced until need for it is felt.

(2) A simplifying of the subject by the elimination of unnecessary material, and by the assumption of many of the simpler principles of geometry of which the student does not feel the need of proof.

(3) An attempt to motivate the teaching of geometry in the secondary school, and to make it function in the life of the student, by utilizing as exercises a large number of applied problems of geometry such as are actually encountered in real life.

(4) Correlation of geometry with algebra, and correlation of geometry with trigonometry by the introduction of simple work with the trigonometric ratios in connection with similar polygons.

(5) Elimination from elementary geometry of proofs of incommensurable cases of theorems by use of the theory of limits.

(6) A grading of geometry, through a rearrangement of the subject matter, which is made possible by abandoning the traditional Greek division of geometry into books and re-grouping the material in chapters.

(7) Use of the suggestive method in the treatment of theorems, the proofs of many of the simpler theorems being left, with suggestions, to the student.

In the revision, the sequence of the subject matter has been changed to some extent, and the material of the twelve chapters combined into eleven. The text has been simplified by reducing somewhat the number of formal theorems, corollaries, and constructions.

Complete proofs of more theorems are given than in the older edition, particularly at the beginning of the course, thus simplifying the subject and at the same time providing the student with more perfect models for his own proofs. Where proofs are provided, they are given in full. In the suggestive treatment of theorems, the character of the suggestions has been improved, the analysis which is extensively used affording the best training in original thinking on the part of the student.

The exercises have been thoroughly revised. Those which were too difficult or which demanded technical knowledge on the part of the student have been eliminated. Many new exercises, both theoretical and practical, have been introduced. The exercises have been placed at more frequent intervals, immediately following the theorems, etc., to which they are related.

New cuts have been made for all of the figures. Also much care has been given to making the page forms as attractive as possible and the entire volume in every way convenient to use.

Grateful acknowledgment of the authors is due to all those who by timely suggestions have aided in the preparation of this volume; especially to Professor H. E. Cobb and Professor A. W. Cavanaugh of Lewis Institute, Chicago; to Miss Alice M. Lord of the High School, Portland, Maine; and to Professor Guido H. Stempel of Indiana University. Special acknowledgment is due also to Mr. Charles McCauley, of Chicago, who has made the excellent illustrations.

JOHN C. STONE,
JAMES F. MILLIS.

JANUARY, 1916.

SUGGESTIONS TO TEACHERS

Chapter I is given for providing a concrete approach to the subject, for acquainting the student with the uses of the tools of geometry, for developing a body of experience and imagery as a basis of demonstrative geometry, and for introducing definitions, axioms, and other fundamental ideas. Impress upon the student the fact that all axioms and other principles printed in italics must be memorized for future use. The amount of time to be devoted to this chapter depends upon the previous training of the students.

Each student should be provided at the beginning of the course with a straightedge, compasses, protractor, and triangle. These instruments may be obtained from supply houses at small cost. The use of these instruments by the student in the accurate constructions of the figures needed throughout the course should be insisted upon. An accurately constructed figure often will suggest the proof of a theorem, while a poorly constructed one often is misleading.

In beginning the study of proofs in Chapter II, teach thoroughly the matter in § 30. Emphasize the dependence of the proof of a theorem upon a knowledge of the definitions, theorems, etc., which have preceded it in the text. Hence, urge the student to memorize, not only definitions, etc., but also the exact wordings of theorems (not the proofs) in the text as they are studied. In the proofs of theorems, the student should be required to give the reason for each step in full, and not make merely a reference to a previous section. In oral proofs, when previous theorems are referred to, they should be quoted exactly.

When studying a theorem demonstrated in the text, encourage students to seek other proofs, if such exist, thus developing originality and independence. Progress slowly with the work, until the nature of a proof is well established.

Three types of proof are employed in every geometry text: (1) simple direct proof; (2) indirect proof; and (3) proof involving the superposition of figures. It is in order to avoid the introduction of all three of these types of proof in the first few theorems that, at the beginning of Chapter II, the present text assumes from observation and measurement the equality of corresponding angles made by a transversal of parallel lines. This leads to an easy sequence of demonstrations by the simple direct method of proof alone. Thus the student encounters and overcomes only one difficulty at a time.

The traditional assumption, in textbooks, of the so-called *Axiom of Parallels — that through a given point only one line can be drawn parallel to a given line* — is believed to be unpsychological, for it leads to a sequence of theorems that require all three types of proof at the beginning of the subject, and in very close succession. This makes geometry so difficult at the outset that a student is apt to become discouraged and get a dislike for the subject.

In the present text, the simple direct proof is introduced in § 27; the indirect proof is not encountered until § 52; while the proof by superposition, which to most students seems the most subtle of all, is not introduced until Chapter III. In addition to being more psychological, the plan of this text is as mathematically sound as that which employs the Axiom of Parallels, for in each case a single mathematical assumption is made. If either assumption is made, the other may be proved. (See Corollary, § 46.)

No student is expected to solve all of the exercises in the text. Select those which are best suited to the powers of the class. Those which are omitted at first may be used later, in connection with reviews, if desired. Have the student study carefully the

"Methods of Attack" given in Chapter VII (§ 149 and the following exercises). Much that is given in this chapter should be presented from time to time by the teacher, as need of it arises in the early exercises. Special study, however, is expected to be given to this matter before taking up the extensive list of miscellaneous exercises for review at the end of Chapter VII.

Teachers are urged to use as many of the applied problems as time will permit. None of them is too technical for the average boy or girl. The construction of instruments and their use in out-of-door problems, suggested at different points of the book, will prove quite practical and of great interest and value to the student. It is through this practical work in geometry that the subject is to be made to function extensively in the life of the student.

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SYMBOLS

$=$, is equal to; equals.
$\equiv$, is identically equal to.
$\sim$, is similar to.
$\cong$, is congruent to.
$>$, is greater than.
$<$, is less than.
$\doteq$, approaches as a limit.
$\parallel$, is parallel to; parallel.
$\perp$, is perpendicular to; perpendicular.
$\therefore$, therefore.
$\cdots$, and so on.
$\angle$, angle.
$\triangle$, triangle.
▱, parallelogram.
$\odot$, circle.

The plural of any symbol representing a noun is obtained by affixing the letter *s*. Thus, $\angle$s represents angles.

ABBREVIATIONS

Ax., axiom.
Alt., alternate.
Comp., complementary.
Cor., corollary.
Corres., corresponding.
Def., definition.
Ex., exercise.
Ext., exterior.
Hyp., hypothesis.
Int., interior.
Rect., rectangle.
Rt., right.
St., straight.
Supp., supplementary.

PLANE GEOMETRY

CHAPTER I

FUNDAMENTAL IDEAS

1. Geometry. — Of all the arts and sciences which man has discovered in the endeavor to meet his practical needs, *geometry* is one of the most ancient. History shows that a knowledge of geometry was used by the Egyptians in land surveying many centuries before the time of Christ. It is said by Herodotus, a Greek historian who traveled in ancient Egypt, that because from time to time the overflow of the Nile washed away the landmarks and some of the land, it became necessary to have the land measured frequently in order to determine the just distribution of taxes. The Egyptians must also have used considerable knowledge of geometry in the construction of the wonderful pyramids, obelisks, and temples. Such rules for practical measurement as the Egyptians knew were the result of centuries of observation and experience. Many Greek scholars traveled in this land of the pyramids and

learned the Egyptian art of measurement. The Greeks subsequently discovered many new truths, and arranged all of them into a system or science, which they called *geometry*, meaning *land* or *earth measurement.*

As civilization has advanced, geometry has come to have a much wider practical use than merely in land measurement or surveying. It is employed in architecture, designing, building, engineering, astronomy, navigation, the construction and use of measuring instruments and machinery, etc.; in short, it is needed in a vast number of the modern activities of man.

2. Solids, surfaces, lines, and points. — Geometry deals with ***solids*, *surfaces*, *lines*, *and points***, and with combinations of these. It is devoted to the consideration of the properties of these elements and their combinations, and to the application of the truths relating to them in practical work.

Any physical object, such as a book, a ball, or a block of stone, occupies *space*. Geometry is not concerned with the substance of the object, but with the properties of the space or room occupied by it.

Thus, the cylindrical object here represented is a *physical solid,* which can be touched or weighed. If one imagined this solid removed, the empty space formerly occupied by it would be of the size and shape represented by the diagram to the right. It is with the latter that geometry deals.

The space occupied by an object is called a **geometric solid.**

Every portion of space, or geometric solid, is separated from the adjacent or neighboring space by a **surface.**

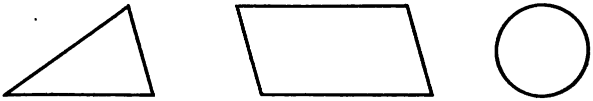
A portion of a surface is separated from the remainder of the surface by a **line.**

A line is separated into two parts by a **point.**

For example, the above geometric solid is separated from the surrounding space by a surface consisting of two flat parts and one round part. The round part of the surface is separated from each flat part by a line.

The chief characteristic of a point is that it has *position.* If a point were imagined to move, it would *describe a line* as its path. Similarly, if a line were to move in any way, except along itself, it would *describe a surface.* And if a surface were to move in any way, except along itself, it would *describe a solid.*

3. Geometric figures. — Any geometric solid, surface, line, point, or combination of them, is called a **geometric figure.**



Thus, the combinations of lines and points here represented, with which the student became familiar in the study of arithmetic, are geometric figures. Each is composed of one or more lines and points, all of which lie in one flat surface.

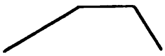
4. Plane surfaces. — A *flat surface,* such as the floor, blackboard, or top of a desk, is called a **plane surface,** or merely a **plane.** *Plane geometry,* to which the following pages are devoted, deals only with those geometric figures which lie in a plane.

5. Representations of points and lines. — Since a *point* has no substance, it is invisible. It may be represented to the eye by a small *dot* or *cross*. It may be named by putting a capital letter near the dot or cross. Thus, we may speak of "point *A*" or "point *B*." Since a *line* has no substance, it is invisible, but may be represented to the eye by some kind of *mark*. It is not the lines nor points, then, that we see, but *the physical representations of them.*

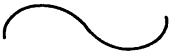
6. Kinds of lines. — Three kinds of lines and combinations of them may be drawn in a plane, namely, *straight lines*, *broken lines*, and *curved lines* or *curves*.

A **straight line** is represented by placing a *ruler*, or *straight-edge*, upon a plane surface and marking along the edge of the ruler.

STRAIGHT LINE



BROKEN LINE



CURVED LINE

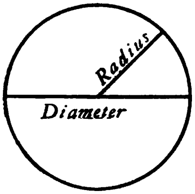
A **broken line** is a line which is not straight, but which is made up of parts all of which are straight.

A **curved line**, or **curve**, is a line no part of which is straight.

The principal curved lines used in elementary geometry are *circles* and *arcs* of circles.

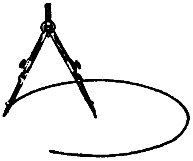
A **circle** is a closed curve all points of which are equally distant from a point within, called the **center**. The distances from all points of a circle to the center are called **radii**, which, therefore, are equal. Two radii in a straight line form a **diameter**.

An **arc** of a circle is a portion of a circle.

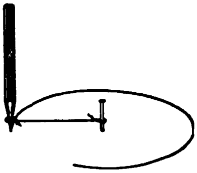


A CIRCLE

7. Drawing and measuring instruments. — Since the time of the ancient Greeks the figures considered in elementary geometry have been only those which may be drawn by the use of two instruments, the **straightedge** (unmarked ruler), and **compasses** (also called **dividers**). The *straightedge*, as pointed out in § 6, is used for drawing straight lines. The *compasses* are used for drawing circles and arcs of circles and for measuring and comparing distances.



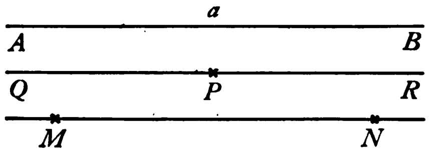
A good substitute for compasses, in case the latter are not accessible, consists of a cord with a loop tied in one end, into which the pencil or crayon may be inserted. The use of the cord in drawing circles or arcs, or in marking off distances, is easily seen.



The uses of various other drawing and measuring instruments, such as are employed in practical work of the present day, will be pointed out farther along in the book.

THE STRAIGHT LINE

8. Line, ray, and line-segment. — While the mark which represents a straight line must be of limited length, a straight line must be thought of as *extending without end* in two opposite directions. A straight line is named by writing

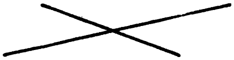


two capital letters near it, as "line *AB*"; or by writing one small letter near it, as "line *a*." Hereafter when the word *line* alone is used, it shall be understood as referring to a *straight* line.

If a point divides a straight line into two parts, each part is called a **ray**. Thus point P divides line QR into the rays PR and PQ. Evidently, a ray has *only one end*, and extends away from that end indefinitely.

The portion of a straight line lying between two points of it is called a **line-segment** or **segment**. Thus MN is a line-segment, with ends at M and N.

9. Properties of straight lines. — Two straight lines that cross at a point are said to **intersect**. Two straight lines in a plane that do not meet at all are called **parallel**.



INTERSECTING LINES

PARALLEL LINES

Mark a point. Draw four straight lines through it. How many straight lines can intersect at one point?

Mark two points, A and B. Draw a straight line passing through both points. Can a second straight line be drawn through these two points?

Mark a point. Draw two straight lines through it. Can these two lines intersect at a second point?

What is the shortest distance between two points?

The properties of straight lines may be summarized as follows:

I. *Any number of straight lines can be drawn through a point.*

II. *One and only one straight line can be drawn through two points. Or, two points determine a straight line.*

It follows that *two straight lines which are placed so that they have two or more points in common form one line.*

III. *Two straight lines cannot intersect in more than one point. Or, two intersecting straight lines determine a point.*

IV. *The length of the line-segment connecting two points is the shortest distance between them.* The segment is taken to represent the **distance** between the points.

EXERCISES

1. Fold and crease a piece of paper to make a straightedge.

2. If the edge of your ruler is placed so that it touches a straight line at two points, will it touch the line all along the ruler? Upon what property of the straight line in § 9 is the answer based?

3. Test the accuracy of your ruler as follows: Mark along the edge. Then reverse the ruler, placing it so that it touches the mark at two points, and draw a second mark. What will show whether the ruler is true or not?

4. Test the straightness of the segment AB as follows: Place over it a piece of thin paper, through which the segment may be seen, and trace it, marking points A' and B' over A and B, respectively. Then reverse the paper so that point A' falls upon point B and point B' upon point A. Is the segment straight? Explain.

5. Explain why a gun or other sighting instrument has two "sights" on it.

6. Explain what is meant by "Two points determine a straight line."

7. Show how the principle in Ex. 6 may be used to set three stakes in a straight line.

8. Explain how the fact that when a cord is stretched taut it is straight, follows from the fourth property of a straight line in § 9.

9. Three points not in a straight line determine how many straight lines? Draw them. Four points no three of which are in a straight line determine how many straight lines? Draw them.

10. Explain what is meant by "Two intersecting straight lines determine a point."

11. How could you locate a point known to be on each of two straight lines?

12. Draw two parallel lines by marking along both edges of a ruler while holding it in one position. What is known about two straight lines which are *not* parallel?

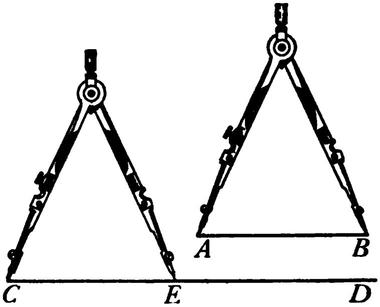
13. Point out four pairs of parallel lines in the room.

10. Construction, measurement, and comparison of line-segments. — Two things are said to **coincide** when they are placed in identically the same position throughout.

Two line-segments are equal if, and only if, they may be made to coincide throughout.

Compasses may be used to mark off equal segments on a line, to construct a segment equal to a given segment, or to compare the lengths of segments.

Thus, to mark off a segment on CD equal to the segment AB, first place one leg of the compasses at A. Then, by adjusting the compasses, make the other leg fall at B. Then, without changing the adjustment, place one leg at C and mark the point E on CD with the other. Then $CE = AB$.



EXERCISES

1. Draw a line and mark off a segment on it equal to a given line-segment.

2. Mark off four equal segments on a line.

3. Draw a line-segment twice as long as a given line-segment.

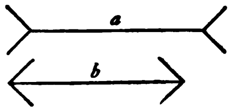
4. Draw a line-segment five times as long as a given line-segment.

5. Draw a line-segment equal to the sum of two given line-segments.

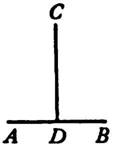
6. Draw a line-segment equal to the sum of three given line-segments.

7. Draw a line-segment equal to the difference between two given line-segments.

8. Draw any two line-segments. Show by the compasses which is the longer.

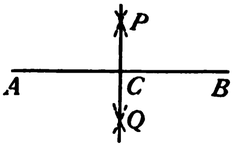


9. Estimate which of the line-segments *a* and *b* is the longer. Check the conclusion by testing with compasses.



10. Estimate which of the line-segments *AB* and *CD* is the shorter. Check the conclusion by use of compasses.

11. **To bisect a line-segment.**—Any geometric magnitude is said to be **bisected** when it is cut into two equal parts.



Any segment *AB* may be bisected as follows: With the compasses conveniently opened, place one leg at *A* and with the other draw arcs on both sides of *AB*. Then, without changing the opening of the compasses, place one leg at *B* and with the other draw arcs cutting the first arcs at *P* and *Q*. Draw the straight line *PQ*, intersecting *AB* at *C*. Then *AB* is bisected at *C*.

Show that *AC* and *CB* are equal by testing them with compasses.

EXERCISES

1. In drawing the arcs to bisect a given line-segment, how far must the compasses be opened?

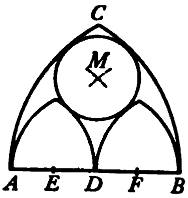
2. Draw several different line-segments and bisect each. Test the accuracy of each construction by means of compasses.

3. Divide a given line-segment into four equal parts. (First bisect it, then bisect each of the parts.)

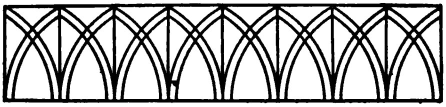
4. Divide a given line-segment into eight equal parts.

5. Can the method of bisecting a line-segment be employed to divide a given line-segment into five equal parts? Into six? Explain. Into how many equal parts in general can a line-segment be divided by this method?

6. This drawing shows the design of a Gothic window. Arc BC is drawn with radius AB and center A. Arc AC is drawn with radius AB and center B. The small arcs are drawn with radii equal to AD and centers at A, D, and B. The center M of the circle is found by drawing arcs with centers A and B and radii equal to AF. Show how to locate the points D, E, and F. Draw a similar design upon any convenient line-segment corresponding to AB.

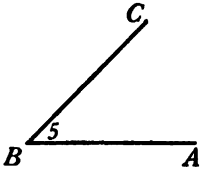


7. Draw a design like the following. First divide a line-segment into eight equal parts. With the ends of these parts as centers, and with a radius equal to two of the parts, draw arcs as shown in the figure. In drawing the smaller arcs use the same centers.



ANGLES

12. An angle. — The figure formed by two rays drawn from the same point is an **angle**. The rays are called the **sides** of the angle and the point is called the **vertex**.



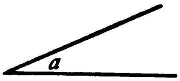
Thus, in the angle here represented, BA and BC are the sides and point B the vertex.

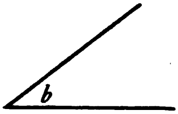
This angle may be named "angle ABC," or "angle B." When using the three letters to name the angle, note that the *letter at the vertex* is written and read in the *middle*.

An angle is sometimes named also by writing a *figure* or a *small letter* within it. Thus the above angle may be named "angle 5."

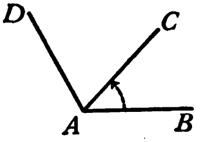
In writing the name of an angle, the symbol $\angle$ is used in place of the word "angle." Thus "angle ABC" is written $\angle ABC$, and "angle a" is written $\angle a$.

13. Size of an angle. — The **size** of an angle does not depend upon the lengths of the marks representing its sides. The sides of an angle are rays, and hence extend without limit from the vertex. Thus ∠ a is less than ∠ b, although their sides are represented by marks of equal lengths.





The size of an angle depends upon the relative *directions* of its sides. It may be measured by the amount of turning of one side about the vertex required to make it fall upon the other side. Thus ∠ BAD is larger than ∠ BAC, because AB would have to be revolved about A farther to fall upon AD than upon AC.

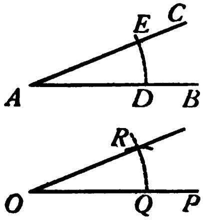


The sizes of two angles may be compared by placing one upon the other so that the vertices are together and a side of one falls along a side of the other, then observing the positions of the other sides. If neither angle may be moved, a trace or copy of one may be made upon thin paper, and this trace applied to the other.

Two angles are equal if, and only if, they may be made to coincide throughout. (See § 10.) *If an angle is placed upon an equal angle so that the vertices and a pair of sides coincide, the other sides must coincide.*

EXERCISE

1. Draw two angles. Name one ∠ ABC and the other ∠ DEF. Cut out or trace ∠ DEF and place it upon ∠ ABC so that E falls upon B and ED falls along BA. If EF falls within ∠ ABC, which angle is greater? What angle shows the difference? If EF falls along BC, what is known of the two angles?

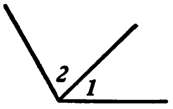


14. To construct an angle equal to a given angle. — An angle may be drawn equal to a given angle as follows:

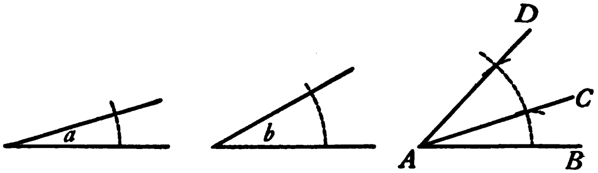
Given ∠ *BAC*. Draw *OP*. Now with the compasses opened conveniently, and with one leg at vertex *A* of the given angle, draw an arc cutting the sides of the given angle at the points *D* and *E*, respectively. Now with one leg at *O*, and without changing the opening of the compasses, draw an arc cutting the ray *OP* at point *Q*. Then place one leg of the compasses at *D* and adjust the compasses until the other leg falls at *E*. Without changing the adjustment, place one leg at *Q*, and with the other draw an arc cutting the first arc at a point *R*. Draw *OR*. Then ∠*POR* = ∠*BAC*.

Test the equality of the angles by the method of § 13.

15. Addition, subtraction, and multiplication of angles. — Two angles having the same vertex and a common side between them are called **adjacent angles**. Thus ∠ 1 and ∠ 2 are adjacent. The *sum* of two angles is obtained by placing them adjacent.



The sum of two angles may be constructed as illustrated



in the figure, by using the construction of an angle equal to a given angle shown in § 14.

Here $\angle BAC$ is drawn equal to $\angle a$. Then $\angle CAD$ is drawn equal to $\angle b$. Hence $\angle BAD = \angle a + \angle b$. By continuing the process three or more angles may be added.

Also angles may be subtracted, or an angle may be multiplied, by means of the construction in § 14.

EXERCISES

1. Draw any angle. Construct an angle equal to it. How may the construction be tested to see if it is accurate?

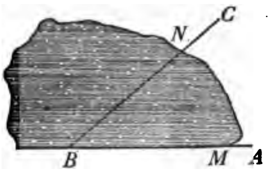
2. Draw any two angles. Construct an angle equal to their sum.

3. Draw any three angles, and construct their sum.

4. Draw two angles, one apparently larger than the other. Construct an angle equal to their difference.

5. Draw an angle. Then construct another angle twice as large.

6. Draw an angle. Then construct another angle three times as large.

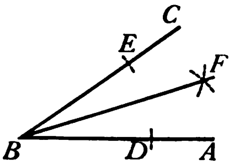


7. An angle may be constructed equal to a given angle by use of a piece of paper with one straight edge, as follows: To construct an angle equal to $\angle ABC$, place the paper with the straight edge along AB, and mark the vertex B on the edge of the paper. Also mark the point N on the opposite edge of the paper where the side BC intersects it. Then $\angle MBN = \angle ABC$. By using $\angle MBN$ another angle equal to $\angle ABC$ may be drawn wherever desired.

Use this method of constructing an angle equal to a given angle.

May this device serve for comparing and testing angles also? Explain.

16. To bisect an angle. — An angle ABC may be bisected (See § 11) as follows: With center at B and any convenient radius, draw arcs cutting AB at D and BC at E. Then with centers at D and E, and equal radii of any convenient length, draw arcs meeting at a point F. Draw the ray from B through F. Then $\angle ABC$ is bisected by BF.



Test $\angle ABF$ and $\angle FBC$ to see if they are equal.

EXERCISES

1. Draw an angle and bisect it. Test to see if the construction is accurate. Repeat until proficient in the construction, and the process is well fixed in mind.

2. Divide a given angle into four equal parts.

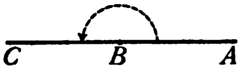
3. Divide a given angle into eight equal parts.

4. Can the process of bisecting an angle be used to divide an angle into five equal parts? Into six? Into how many equal parts can an angle be divided by this method?

5. Draw two intersecting lines. Bisect each of the four angles thus formed. What is observed about these bisectors?

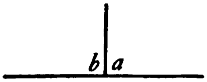
17. **Classification of angles.** — An angle whose sides extend in opposite directions so that they form one straight line is called a **straight angle**, as $\angle ABC$.

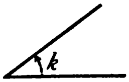
It is evident that the entire angular magnitude around a point is the sum of two straight angles. It is called a **perigon.**

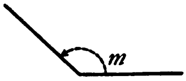


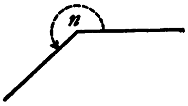
A **right angle** is one half of a straight angle, as $\angle a$ or $\angle b$.

It follows that a right angle may be constructed by bisecting a straight angle. Explain the process.









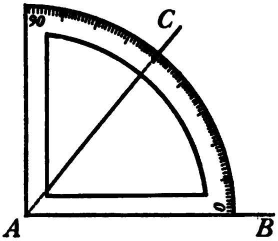
An **acute angle** is an angle less than a right angle, as $\angle k$.

An **obtuse angle** is an angle greater than a right angle but less than a straight angle, as $\angle m$.

A **reflex angle** is an angle greater than a straight angle but less than two straight angles, as $\angle n$.

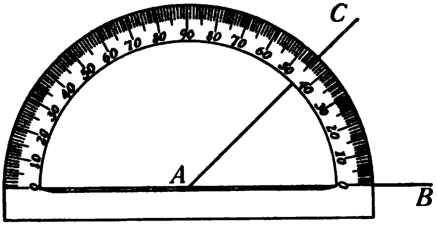
18. The measurement of angles. — If a right angle is divided into 90 equal parts, each of these parts is called a **degree**, which is the principal unit of measure of angles. Thus, a right angle contains 90 degrees, written 90°. A straight angle contains 180°. A perigon, or the entire angular magnitude around a point, contains 360°.

A degree is divided into 60 **minutes** (written 60′), and a minute into 60 **seconds** (written 60″).



A QUADRANT

The **quadrant** and the **protractor** are instruments used for measuring the number of degrees in an angle. To use either instrument for measuring ∠ *BAC*, the center is placed over the vertex *A* of the angle, and the instrument turned until the zero mark of the scale falls on the side *AB*. Then the point where the side *AC* crosses the scale shows the number of degrees in the angle.



A PROTRACTOR

EXERCISES

1. Draw any angle. Then measure the number of degrees in it with a protractor. Explain the method.

2. Measure ∠*ABC* in § 16 with the protractor. Measure ∠*ABF* and ∠*FBC* in the same figure. Are the latter equal?

3. Draw any angle and bisect it as in § 16. Test the accuracy of the construction by means of the protractor.

4. Draw an angle and bisect it by means of the protractor.

5. Draw any angle. Then draw another angle equal to it by means of the protractor.

6. By using the protractor, construct a right angle.

7. Draw angles of 15°, 30°, 45°, 60°, 72°, 120°, 165°.

8. How many degrees between the hands of a clock at 1 o'clock? At 3 o'clock? At 6 o'clock? At 9 o'clock? At 12 o'clock?

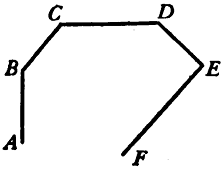
9. One angle contains 64° 24′ 35″ and another contains 47° 28′ 42″. Find their sum.

10. Subtract 23° 48′ 50″ from 90°. From 62° 25′.

11. This table contains the field notes of the survey of the line *ABCDEF*, beginning at *A*, by surveyors. The map of the survey is drawn to scale, 800 ft. being represented by 1 in. (The table means that the point *B* is due north from *A*, 300 ft.; the point *C* is 40° east of north from *B*, 250 ft.; etc.)

POINTS	DIRECTIONS	DISTANCES
B	N.	300 ft.
C	N. 40° E.	250 ft.
D	E.	400 ft.
E	S. 45° E.	255 ft.
F	S. 42° W.	500 ft.

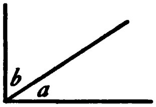
Draw a copy of the map, using a scale of 200 ft. to an inch. Use the protractor in turning off the angles at *B*, *C*, etc.

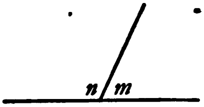


12. These are the notes of a survey of a river bank. Make a drawing to scale, showing the course of the river, and mark the positions of the objects on it.

POINTS	DIRECTIONS	DISTANCES
Rail Fence	N.	800 ft.
Mouth Creek	N. 24° W.	2400 ft.
Old Mill	N. 35° E.	1000 ft.
Poplar Tree	N. 20° E.	600 ft.
Bridge	E.	2000 ft.
Schoolhouse	S. 80° E.	1600 ft.
Stone Wall	N. 45° E.	2000 ft.

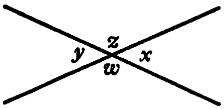
19. Complementary, supplementary, and vertical angles. — Two angles whose sum equals a right angle are called **complementary**, or one angle is the **complement** of the other, as $\angle a$ and $\angle b$.





Two angles whose sum equals a straight angle are called **supplementary**, or one angle is the **supplement** of the other, as $\angle m$ and $\angle n$.

Two intersecting straight lines form four angles with a common vertex. Of these angles two that are not adjacent are called **vertical angles**, as $\angle x$ and $\angle y$.



How many pairs of vertical angles are formed by two intersecting straight lines? Name another pair in the figure.

EXERCISES

1. What is the complement of an angle of 60°? Of 45°? Of 72°?

2. What is the complement of an angle of 48° 24′ 15″? Of 62° 48″?

3. Construct the complement of a given angle ABC by use of straightedge and compasses. By use of straightedge and protractor.

4. What is the supplement of an angle of 120°? Of 80°? Of 45°?

5. What is the supplement of an angle of 93° 40′ 30″? Of 133° 45″?

6. Construct the supplement of a given angle ABC.

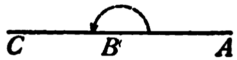
7. What is the angle which equals 5 times its complement? (Form an equation.)

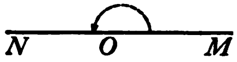
8. What is the angle which equals 8 times its supplement?

9. Two angles are complementary. Three times the smaller is 5° less than twice the larger. What are the angles?

10. In the figure of vertical angles, § 19, if $\angle x = 60°$, how many degrees are there in $\angle z$? Then how many degrees in $\angle y$? Then how many degrees in $\angle w$?

20. Equality of special angles. — Two straight angles may be compared as to size by the method of § 13, that is, by placing one upon the other. If straight angle MON is placed upon straight angle ABC so that O falls upon B and OM falls along BA, it follows from § 9, II, that ON must fall along BC. Hence:

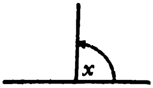


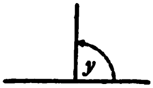


I. *Any two straight angles are equal.*

If $\angle x$ and $\angle y$ are right angles, each is one half of a straight angle. Show the straight angles in the figure.

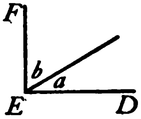
Since a right angle is one half of a straight angle, and halves of equals are equal, it follows from I that:

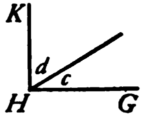




II. *Any two right angles are equal.*

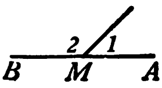
If $\angle a$ and $\angle b$ are complements, and $\angle c$ and $\angle d$ are complements, what kind of angles are $\angle DEF$ and $\angle GHK$? What is the relation of $\angle DEF$ to $\angle GHK$? Since equals subtracted from equals leave equals, if $\angle a$ and $\angle c$ are equal, what is the relation of $\angle b$ and $\angle d$? Hence:

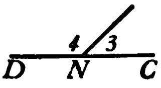




III. *The complements of equal angles are equal.*

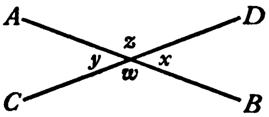
If $\angle 1$ and $\angle 2$ are supplements, and $\angle 3$ and $\angle 4$ are supplements, what kind of angles are $\angle AMB$ and $\angle CND$? What is the relation of $\angle AMB$ to $\angle CND$? Since equals subtracted from equals leave equals, if $\angle 1$ and $\angle 3$ are equal, what is the relation of $\angle 2$ and $\angle 4$? Hence:





IV. *The supplements of equal angles are equal.*

In the figure $\angle x$ and $\angle y$ are vertical angles. Show why $\angle x$ and $\angle z$ are supplementary. Show why $\angle y$ and $\angle z$ are supplementary. Since both $\angle x$ and $\angle y$ are supplements of $\angle z$, what must be the relation of $\angle x$ and $\angle y$? Why? Hence:



V. *Vertical angles are equal.*

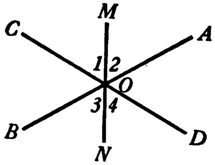
EXERCISES

1. If $\angle x$, in the figure of vertical angles, § 20, is 60°, how many degrees are there in each of the other three angles? Explain.

2. If one of the four angles formed by two intersecting straight lines is 90°, or a right angle, what is each of the other three angles? Explain.

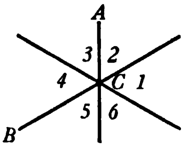
3. If three straight lines meet at one point, how many angles are formed? Of these angles how many is it necessary actually to measure with the protractor in order to determine the sizes of all? Draw a figure and explain.

4. The lines AB and CD form two vertical angles, $\angle AOC$ and $\angle BOD$. MN is drawn through O, making $\angle 1 = \angle 2$. Compare $\angle 1$ and $\angle 4$; also $\angle 2$ and $\angle 3$. Then since things equal to the same thing or equal things are equal, what must be the relation of $\angle 3$ and $\angle 4$? How else may this conclusion be expressed, using the term "bisector"?

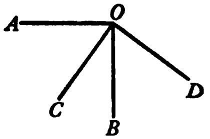


5. If lines are drawn through a point on a straight line, making all of the angles on one side of that line equal, what relation must exist between the angles on the other side of the line? Why? Draw a figure to illustrate.

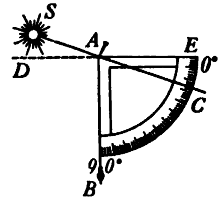
6. If three straight lines meet at one point, show that the sum of any three angles, no two of which are adjacent, as $\angle 1$, $\angle 3$, and $\angle 5$ in the figure, is equal to a straight angle.



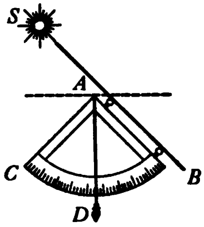
7. In the same figure, show that $\angle 1 + \angle 3 = \angle ACB$.



8. In the adjoining figure, ∠ *AOB* and ∠ *COD* are right angles. Explain why ∠ *AOC* must equal ∠ *BOD*.



9. The angle of elevation of the sun above the horizon, as ∠ *SAD*, may be measured as follows: A quadrant is held in a vertical position, so that a plumb line *AB*, which is suspended from a pin at *A*, falls upon the 90° mark. The shadow of the pin falls upon the scale at *C*, which shows the angle of elevation of the sun. Explain why.



10. *Tycho Brahe* (1546–1601), a Danish noble who built and operated the first real astronomical observatory, constructed and used a big quadrant for measuring the altitudes of the stars, or their angles of elevation above the horizon. When the instrument was held in a vertical position and the sights *A* and *B* aligned with the star *S*, the angle of elevation of the star was determined by reading ∠*CAD*. Explain why.

NOTE. — It will be found extremely practicable and interesting for students to make such instruments as those described in Ex. 9 and Ex. 10, and to use them in making actual measurements. They may be used in many out-of-door problems, such as determining the latitude of the school by measuring the altitude of the North Star, determining the heights of objects, etc. Equally good instruments may be devised by tacking protractors on boards and suspending plumb lines from the centers.

AXIOMS

21. Axioms. — In the reasoning in § 20 and in some of the exercises which followed, use was made of such general principles as "Halves of equals are equal," "Equals substracted from equals leave equals," and "Things which are equal to the same thing, or to equal things, are equal to each other." These principles are called *axioms*.

An **axiom** is a statement about quantities in general which is assumed to be true without proof.

The student is already familiar with a number of axioms used in algebra. The following axioms are given here for future reference:

I. *Things which are equal to the same thing, or to equal things, are equal to each other.*

II. *If equals are added to equals, the sums are equal.*

III. *If equals are subtracted from equals, the remainders are equal.*

IV. *If equals are multiplied by equals, the products are equal.*

V. *If equals are divided by equals, the quotients are equal.*

VI. *Like powers, or like positive roots, of equals are equal.*

VII. *If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by the same positive number, the results are unequal in the same order.*

VIII. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

IX. *If unequals are added to unequals in the same order, the sums are unequal in that order.*

X. *The whole of a thing is equal to the sum of all of its parts, and is greater than any one of its parts.*

XI. *If the first of three things is greater than the second, and the second greater than the third, then the first is greater than the third.*

XII. *A quantity may be substituted for its equal in any expression.*

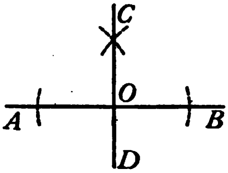
PERPENDICULAR LINES

22. Perpendicular lines. — If two intersecting straight lines form right angles, the lines are **perpendicular** to each other. When one line is drawn perpendicular to another, the point of intersection is called the **foot** of the perpendicular.

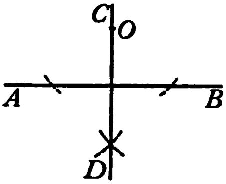
In written work the symbol ⊥ often is used in place of the words "perpendicular," "perpendicular to," or "is perpendicular to." Thus in the figure of the next section, $CD \perp AB$.

The length of the perpendicular line-segment from an external point to a line is called the **distance** from the point to the line.

23. To draw a perpendicular to a given line at a given point. — Since perpendicular lines meet at right angles, the process of drawing a line *CD* perpendicular to a given line *AB* at a given point *O* on *AB* is the same as the process of constructing a right angle *BOC*. Explain the complete process.



24. To draw a perpendicular to a given line, through a given external point. — In order to draw a line *CD* perpendicular to a given line *AB*, through a given point *O* not on *AB*, proceed as follows: With any convenient radius and center at *O*, draw arcs intersecting *AB* at two points. With these two points as centers and equal radii of any convenient length, draw two arcs that intersect. Draw a line through *O* and this point of intersection. This will be the required line. Point out the steps of the construction in the adjoining diagram.



EXERCISES

1. Observe that in each of the constructions in § 23 and § 24, one point O of the required perpendicular line CD was given, and only one other point was located through which to draw it. Of what property of the straight line in § 9 is this an application?

2. In the construction of § 24, to draw the arcs with center O what length of radius must be taken?

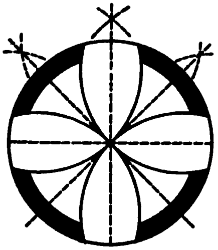
3. Mark points on three different lines, and draw perpendiculars to the lines at those points. Test the accuracy with a protractor.

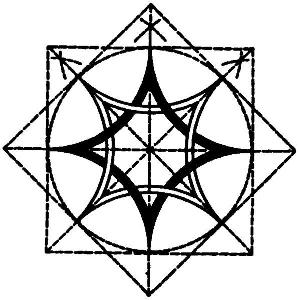
4. Draw a perpendicular to a given line, through a given external point. Repeat until the process is thoroughly fixed in mind. Test the accuracy with a protractor.

5. Mark two points on a straight line. Then draw a perpendicular to the line at each point. What is the relation of the two perpendiculars?

6. Draw a diameter of a circle. Draw a second diameter perpendicular to the first.

7. Study the constructions of the following designs, and see how the method of drawing perpendicular diameters in Ex. 6 is employed in their constructions. Draw designs similar to these, but larger.

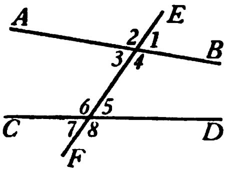




CHAPTER II

PROOFS: PARALLEL AND PERPENDICULAR LINES

25. A transversal of lines. — A straight line which intersects two or more straight lines is called a **transversal** of those lines. Thus EF is a transversal of AB and CD in the figure.



A transversal of two lines makes four angles at each intersection, as ∠ 1, ∠ 2, ∠ 3, ∠ 4, and ∠ 5, ∠ 6, ∠ 7, and ∠ 8.

∠ 3, ∠ 4, ∠ 5, and ∠ 6 are called **interior** angles, and ∠ 1, ∠ 2, ∠ 7, and ∠ 8 are called **exterior** angles.

Two angles such as ∠ 1 and ∠ 5 are called **corresponding** angles. How many pairs of corresponding angles are there? Name them.

Two angles such as ∠ 3 and ∠ 5 are called **alternate interior** angles. How many pairs of alternate interior angles are there? Name them.

Two angles such as ∠ 2 and ∠ 8 are called **alternate exterior** angles. How many pairs of alternate exterior angles are there? Name them.

Two angles such as ∠ 4 and ∠ 5 are called **consecutive interior** angles. Name another pair of consecutive interior angles.

Two angles such as ∠ 1 and ∠ 8 are called **consecutive exterior** angles. Name another pair of consecutive exterior angles.

EXERCISES

1. In the figure of § 25, what angles are equal? Why?

2. What angles in the same figure are supplementary?

26. Corresponding angles compared. — By marking along both edges of a ruler, draw two parallel lines. Draw any transversal of these lines. Compare any two of the corresponding angles by tracing one on thin paper and applying the trace to the other, by the method suggested in Ex. 7, § 15, or by measuring the angles carefully with a protractor. Repeat the comparison by drawing a new figure. What is the conclusion? Hence we infer the following general truth:

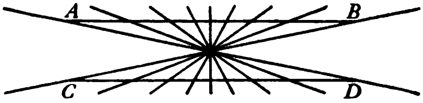
*If two parallel lines are cut by a transversal, the corresponding angles are equal.**

27. Establishing truth by reasoning. — In the preceding pages a number of geometric figures have been drawn and the accuracy of the constructions tested by measurement, tracing, etc. Also it has been seen how a geometric truth, such as that in § 26 above, may be discovered and proved by means of measurements and various kinds of practical tests. This, however, is a slow, and often unsafe, method of testing the accuracy of the constructions of figures or of discovering or establishing general truths. It is better, if possible, to arrive at mathematical truth through a *a course of reasoning* alone. In the subsequent sections we shall see how this is done. The following exercises show the danger of drawing conclusions from the mere appearances of geometric figures.

EXERCISES

1. Review Ex. 9 and Ex. 10, page 9.

2. Judge whether AB and CD in this figure are both straight lines. Then test each line by applying a ruler to it.



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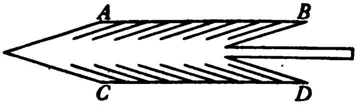
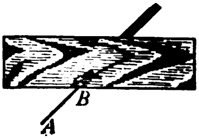
*To THE TEACHER. — See *Suggestions to Teachers*, page vii.

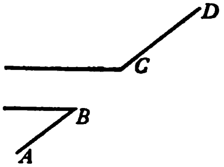
3. Is the width between A and C of the arrow head the same as between B and D? Test with compasses.



4. Which of the lines at the top of the figure is a prolongation of AB? Test by applying a ruler.



5. Are the sides AB and CD of these angles in one straight line? Test by applying a ruler.



28. A theorem.—The statement of a geometric truth to be proved is called a **theorem.**

Thus, the statement in § 26, that "*If two parallel lines are cut by a transversal, the corresponding angles are equal,*" is a theorem.

Every theorem consists of two parts, the *hypothesis* and the *conclusion.* The **hypothesis** consists of the facts expressed which are *supposed* or *known* to be true. The **conclusion** consists of the statement of the truth or truths *to be proved.*

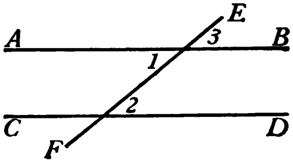
For example, in the theorem stated above, the hypothesis is that two lines are parallel, and that they are cut by a transversal, forming corresponding angles. The conclusion is that the corresponding angles are equal.

A course of reasoning by which the conclusion is shown to follow from the hypothesis is called the **proof** or **demonstration** of the theorem.

The theorems which follow in the book will be *proved by pure reasoning.*

Observe the character of the reasoning in the proof of the following theorem.

29. Theorem. — *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Hypothesis. *AB* is parallel to *CD*, and transversal *EF* forms the alternate interior angles 1 and 2.

Conclusion. $\angle 1 = \angle 2$.

Proof. 1. *AB* is parallel to *CD*. By hypothesis

2. Therefore $\angle 3 = \angle 2$. By § 26

(If two parallel lines are cut by a transversal, the corresponding angles are equal.)

3. But $\angle 1 = \angle 3$. By § 20, V

(Vertical angles are equal.)

4. Therefore $\angle 1 = \angle 2$. By Axiom I

(Things which are equal to the same thing, or to equal things, are equal to each other.)

Study the details of the proof above carefully. Then close the book and see if you can draw the figure and write out the theorem and complete proof, including the hypothesis, conclusion, and all steps of the reasoning. This method should be used in studying all theorems of the book of which the proofs are given in full.

30. Essential points of a proof. — As illustrated in the proof of § 29, the hypothesis and conclusion of a theorem should be expressed concretely, in terms of the particular figure that is drawn. Thus, the conclusion is not stated, "The alternate interior angles are equal," but "$\angle 1 = \angle 2$."

The steps in the reasoning are numbered 1, 2, 3, etc., for clearness and convenience of reference.

Each statement of a truth in the reasoning *must be supported by a reason.*

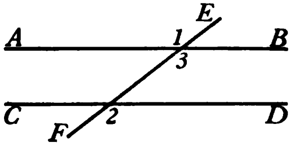
The reason for a step is always some *known truth, which has been presented before*, either as (1) an *axiom*, (2) a *definition*, (3) a *theorem or other statement of a truth or geometric principle* which has been proved or otherwise accepted as true, or (4) the *hypothesis* of the theorem being proved.

In the proofs of the text that follow, the *reasons* are given in the form of references, as to § 26, to Ax. I, etc., printed in the right-hand margin. This is not only for saving space, but also for allowing the student a chance to think of the reasons for himself. In studying the text, he may test the accuracy of his reasoning by looking up the references when necessary. The student will find it wise to memorize all axioms, definitions, theorems, etc., which may be used as reasons in the proofs of theorems, to avoid having to look them up when needed. In all proofs, written or oral, the student should *give the reasons in full.* Reasons *written* out in full should be put in parentheses, under the steps to which they apply, as in § 29. These should be made as brief as possible, however, by means of symbols and abbreviations. Thus, in § 29 the reason for step 3 may be written:

(*Vert.* ∡ *are equal.*)

For conciseness, all of the statements of a written proof should be abbreviated as much as possible, and use made of symbols, as illustrated in the next theorem. For example, the symbol ∥ is used for "parallel," or "is parallel to." Thus "AB is parallel to CD" is written $AB \parallel CD$. For the word "therefore" the symbol ∴ is used. For a complete list of abbreviations and symbols used in geometry, see page x.

31. Theorem. — *If two parallel lines are cut by a transversal, the alternate exterior angles are equal.*



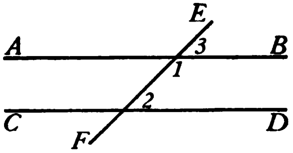
Hypothesis. $AB \parallel CD$, and EF cuts AB and CD, forming alternate exterior angles 1 and 2.

Conclusion. $\angle 1 = \angle 2$.

Proof.	1. $AB \parallel CD$.	Hyp.
	2. $\therefore \angle 3 = \angle 2$.	§ 26
	3. But $\angle 1 = \angle 3$.	§ 20, V
	4. $\therefore \angle 1 = \angle 2$.	Ax. I

With the book closed, write out the complete proof.

32. Theorem. — *If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.*



Hypothesis. $AB \parallel CD$, and EF cuts AB and CD, forming consecutive interior angles 1 and 2.

Conclusion. $\angle 1$ and $\angle 2$ are supplementary.

Proof.	1. $AB \parallel CD$.	Hyp.
	2. $\therefore \angle 3 = \angle 2$.	§ 26
	3. $\angle 1 + \angle 3 =$ st. $\angle$.	Def. st. $\angle$
	4. $\therefore \angle 1 + \angle 2 =$ st. $\angle$.	Ax. XII
	5. $\therefore \angle 1$ and $\angle 2$ are supplementary.	Def. supp. $\angle$s

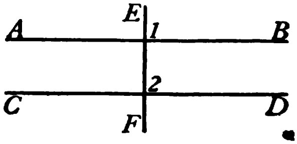
Write out the complete proof without the book.

33. Theorem. — *If two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.*

The proof of this theorem is left to the student. Write it out in full as in § 32.

Suggestions. — Draw a figure and name two consecutive exterior angles $\angle 1$ and $\angle 2$. Name the angle $\angle 3$ that corresponds to $\angle 1$. Compare $\angle 1$ and $\angle 3$, also $\angle 2$ and $\angle 3$. By what axiom may the relation of $\angle 1$ and $\angle 2$ then be obtained?

34. Theorem. — *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Hypothesis. $AB \parallel CD$, and $EF \perp AB$.

Conclusion. $EF \perp CD$.

Proof.	1.	$AB \parallel CD$ and $EF \perp AB$.	Hyp.
	2.	$\therefore \angle 1 = \angle 2$.	§ 26
	3.	And $\angle 1 =$ rt. $\angle$.	Def. $\perp$
	4.	$\therefore \angle 2 =$ rt. $\angle$.	Ax. I
	5.	$\therefore EF \perp CD$.	Def. $\perp$

Write out the complete proof without the book, giving reasons in full.

EXERCISES

1–5. Draw figures and give orally the proofs of the theorems in §§ 29, 31, 32, 33, and 34, stating the reasons in full instead of giving references.

6. Prove the theorem in § 29, using the other pair of alternate interior angles.

7. Prove the theorem in § 31, using the other pair of alternate exterior angles.

8. Prove the theorem in § 32, using the other pair of consecutive interior angles.

9. Prove the theorem in § 33, using the other pair of consecutive exterior angles.

10. Give orally the proof of the theorem in § 29 by referring to an imaginary figure (without drawing it).

11. Give orally the proof of the theorem in § 31 by referring to an imaginary figure.

12. Prove similarly the theorems in § 32, § 33, and § 34.

13. If a transversal cuts two parallel lines, and one of the angles formed is 60°, find the number of degrees in each of the other seven angles. Explain by a diagram.

14. Draw a diagram to represent the rails of two intersecting straight railroad tracks. Point out all equal angles in the figure. Explain.

15. In the figure of § 32, if $\angle 3 = \frac{1}{2}\angle 1$, how many degrees are there in each of the eight angles?

16. If a transveral cuts two parallel lines, making one of two consecutive interior angles equal to three times the other, find the number of degrees in each of the eight angles.

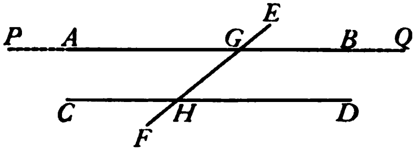
35. Converse theorems. — When the *hypothesis* and *conclusion* of one theorem are the *conclusion* and *hypothesis*, respectively, of another, each theorem is called the **converse** of the other. Thus, the converse of the theorem in § 26 is, "*If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.*"

State the converses of the theorems in § 31, § 32, and § 33.

If a theorem is true, its converse is not necessarily true. For example, the converse of the statement "*All right angles are equal*" is "*All equal angles are right angles*," which is *not* true.

In the following sections the converses of all of the preceding theorems on parallel lines are proved.

36. Theorem. — *If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.*



Hypothesis. AB and CD are cut by EF at G and H, respectively, making $\angle BGE = \angle DHG$.

Conclusion. $AB \parallel CD$.

Proof. 1. Suppose PQ drawn through G parallel to CD.

2. Then $\angle QGE = \angle DHG$. § 26

3. But $\angle BGE = \angle DHG$. Hyp.

4. $\therefore \angle QGE = \angle BGE$. Ax. I

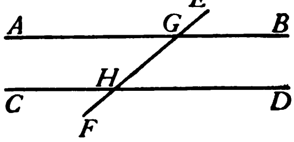
5. $\therefore PQ$ and AB must coincide. § 13

6. Hence, since $PQ \parallel CD$, AB, which coincides with PQ, must be parallel to CD.

Draw a figure and give the proof in full orally. Write it.

Note. — The line PQ, drawn to aid in the proof of this theorem, is called an **auxiliary line**. Such lines, not given in the hypothesis, are used frequently in the proofs of theorems. *They are always denoted by dotted lines.*

37. Theorem. — *If two straight lines are cut by a transversal, making the alternate interior angles equal, the lines are parallel.*



Hypothesis. AB and CD are cut by EF at G and H, respectively, making $\angle AGH = \angle DHG$.

Conclusion. $AB \parallel CD$.

Proof. 1. $\angle AGH = \angle DHG$. Hyp.
2. $\angle AGH = \angle BGE$. § 20, V
3. $\therefore \angle BGE = \angle DHG$. Ax. I
4. $\therefore AB \parallel CD$. § 36

38. Theorem. — *If two straight lines are cut by a transversal, making the alternate exterior angles equal, the lines are parallel.*

Hypothesis. AB and CD are cut by EF at G and H, respectively, making $\angle BGE = \angle CHF$.

Conclusion. $AB \parallel CD$.

Proof. The proof is left to the student. Draw a figure and letter it as indicated in the hypothesis. Compare $\angle AGH$ to $\angle BGE$, and $\angle DHG$ to $\angle CHF$, and then show the relation of $\angle AGH$ to $\angle DHG$. By what theorem does the conclusion then follow?

39. Theorem. — *If two straight lines are cut by a transversal, making the consecutive interior angles supplementary, the lines are parallel.*

Hypothesis. AB and CD are cut by EF at G and H, respectively, making $\angle HGB$ and $\angle DHG$ supplementary.

Conclusion. $AB \parallel CD$.

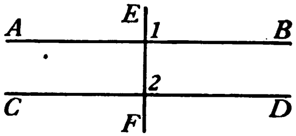
Proof. 1. $\angle HGB$ and $\angle DHG$ are supp. Hyp.
2. $\angle HGB + \angle BGE =$ st. $\angle$. Def. st. $\angle$
3. $\therefore \angle HGB$ and $\angle BGE$ are supp. Def. supp. $\angle$s
4. $\therefore \angle BGE = \angle DHG$. § 20, IV
5. $\therefore AB \parallel CD$. § 36

Draw a figure, lettering it as indicated in the hypothesis, and write the proof in full with the book closed.

40. Theorem. — *If two straight lines are cut by a transversal, making the consecutive exterior angles supplementary, the lines are parallel.*

The proof is left to the student. Draw a figure, lettering it like that in § 39, and use $\angle BGE$ and $\angle FHD$. Show, by reasoning similar to that in § 39, that $\angle BGE = \angle DHG$. By what theorem does the conclusion then follow? Write out the complete proof.

41. Theorem. — *If two straight lines are perpendicular to the same straight line, they are parallel.*



Hypothesis. $AB \perp EF$ and $CD \perp EF$.

Conclusion. $AB \parallel CD$.

Proof. Left to the student. Use § 36. Write the proof in full.

EXERCISES

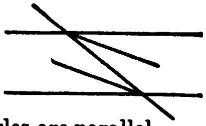
1–5. Draw figures and give orally the proofs of the theorems in §§ 37, 38, 39, 40, and 41. Use other letters in the figures than those of the book.

6. Prove the theorem in § 36 when $\angle AGH = \angle CHF$.

7–9. Prove the theorems in §§ 37, 38, and 39 by drawing auxiliary lines, and proceeding as in § 36.

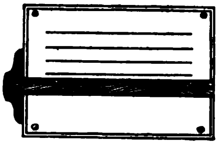
10. Prove that if a line is drawn through a point on one side of a right angle, parallel to the other side, it is perpendicular to the side through which it is drawn.

SUGGESTION. — Use § 34.



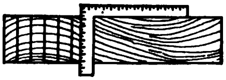
11. Prove that if two parallel lines are cut by a transversal, the bisectors of the alternate interior angles are parallel.

SUGGESTION. — Use the adjoining figure. In order to apply § 37, what angles of this figure must first be proved equal?



12. If two parallel lines are cut by a transversal, what relation exists between the bisectors of the corresponding angles? Prove it.

13. What principle is used in ruling parallel lines on a drawing board by means of a T-square?

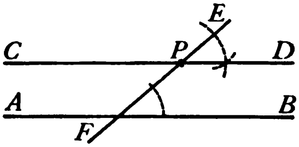


14. What geometric principle does a carpenter use when he marks off parallel lines on a board by moving one blade of his square along an edge of the board and marking along the other?

42. Construction of figures. — The theorems in §§ 36, 37, and 38 suggest convenient methods of drawing parallel lines, one of which is shown in the following section.

In a discussion of a problem of construction there are four steps: (1) the statement of what is *given*, (2) the statement of what is *required* (something to be done), (3) the steps of the *construction*, and (4) the *proof* that the construction is correct. These are illustrated in the following problem.

43. Construction. — *To draw a line through a given point parallel to a given line.*



Given the line AB and the point P not on the line.

Required to draw a line through P parallel to AB.

Construction. 1. Draw any straight line EF through P, meeting AB at F.

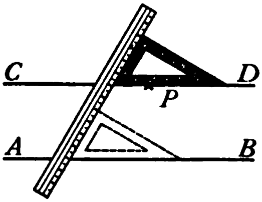
2. Draw CD through P, making $\angle DPE = \angle BFP$.

3. Then CD is the required line.

Proof. 1. $\angle DPE = \angle BFP$. Construction

2. $\therefore CD \parallel AB$. § 36

The foregoing construction may be made by use of a *draftsman's triangle*, instead of compasses, as follows:



Place the triangle with one side along the given line *AB*. Then adjust a ruler against another side of the triangle. Holding the ruler in this position, slide the triangle along it until the side that was along *AB* comes to the given point *P*. Then draw the line *CD* along this side.

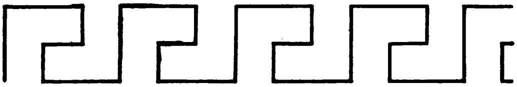
Show that this construction is equivalent to the foregoing.

EXERCISES

1. By use of compasses and straightedge, draw a line through a given point parallel to a given line. Repeat the process until it is thoroughly mastered.

2. By use of a triangle and ruler, draw a line through a given point parallel to a given line. Repeat the construction until proficient.

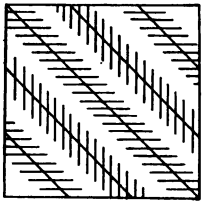
3. Make a large triangle of wood for use in drawing parallel lines on the blackboard. Practice using it at the blackboard.



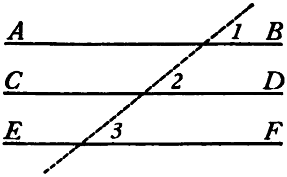
4. The Greek design above is called a *meander*. Make a copy of it. How are the parallel lines drawn?

5. In the figure there are five parallel diagonal lines. Do they look parallel?

Construct a large figure like this by first drawing the five parallel diagonal lines, and see if your figure has the same deceptive appearance.



44. Theorem. — *Two straight lines parallel to the same straight line are parallel to each other.*



Hypothesis. $AB \parallel EF$ and $CD \parallel EF$.

Conclusion. $AB \parallel CD$.

Suggestions. Draw a transversal, forming corresponding angles $\angle 1$, $\angle 2$, $\angle 3$, as in the figure. AB will be parallel to CD if what can be shown? The latter will be true if it can first be shown that $\angle 1 = \angle 3$ and $\angle 2 = \angle 3$. Why? Hence begin by proving these equalities.

Write the proof in full.

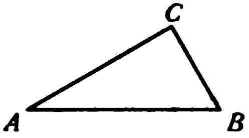
45. A corollary. — A **corollary** is a statement of a truth that is easily deduced from another truth. The following is a corollary of the theorem in § 44.

46. Corollary. — *Only one straight line can be drawn through a given point parallel to a given straight line.*

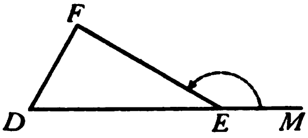
For, if more than one could be drawn, they would be parallel by § 44, which is not possible because intersecting lines are not parallel.

47. A triangle. — If three points not in a straight line are connected by three line-segments, the figure formed is a **triangle**. The three points are called the **vertices** of the triangle, and the line-segments are called the **sides**.

A triangle is named by naming its vertices. The word "triangle" is represented in written work by the symbol △.



TRIANGLE

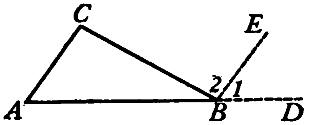


EXTERIOR ANGLE OF TRIANGLE

Thus the triangle above is named "triangle ABC," and may be written $\triangle ABC$.

The angle MEF, formed by the side EF and a prolongation of the side DE through E, is called an **exterior angle** of triangle DEF. The angles formed by the sides themselves are called **the angles**, or, for distinction, the **interior angles**, of the triangle.

48. Theorem. — *The sum of the three angles of any triangle is equal to a straight angle.*



Hypothesis. $\angle A$, $\angle B$, and $\angle C$ are the angles of $\triangle ABC$.

Conclusion. $\angle A + \angle B + \angle C =$ st. $\angle$.

Proof. 1. Produce AB through B to D, and draw $BE \parallel AC$, forming $\angle 1$ and $\angle 2$, as in the figure.

2. Then $\angle A = \angle 1$. § 26

3. And $\angle C = \angle 2$. § 29

4. But $\angle 1 + \angle 2 + \angle B =$ st. $\angle$. Def. st. $\angle$

5. $\therefore \angle A + \angle B + \angle C =$ st. $\angle$. Ax. XII

Draw a figure, and write out the complete proof.

NOTE. — Pythagoras, the famous Greek philosopher and mathematician, knew this theorem about 500 B.C. It is one of the most important and practical theorems of geometry.

49. Corollary 1. — *If two triangles have two pairs of equal angles, the third pair of angles must be equal.*

The proof is left to the student.

50. Corollary 2. — *In any triangle there must be at least two acute angles.*

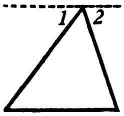
The proof is left to the student.

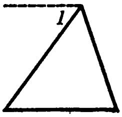
51. Corollary 3. — *An exterior angle of any triangle is equal to the sum of the two opposite interior angles, and hence is greater than either one of them alone.*

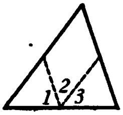
SUGGESTION. — Prove $\angle DBC = \angle A + \angle C$. Write the proof in full.

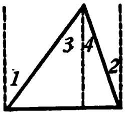
EXERCISES

1–4. Prove the theorem in §48 by the methods suggested by the following figures:









5. If the three angles of a triangle are equal, show that each contains 60°.

6. If one angle of a triangle is 37° 40′ and the other two angles are equal, find the size of each of the equal angles.

7. If one angle of a triangle is 64° 15′ and another is 45° 30′ 30″, find the third angle.

8. If one angle of a triangle is a right angle, what is the sum of the other two angles? Explain.

9. Show that if one angle of a triangle is a right angle, the other two angles must be complementary.

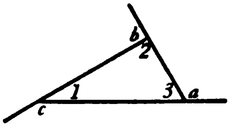
10. Of the three angles of a triangle the second is twice as large as the first and the third three times as large as the first. How many degrees are there in each? (Form an equation.)

11. If one angle of a triangle is a right angle, and one of the acute angles is four times the other, how many degrees are there in each?

12. In a certain triangle an exterior angle is twice the adjacent interior angle, and the two opposite interior angles are equal. How many degrees are there in each of the angles of the triangle?

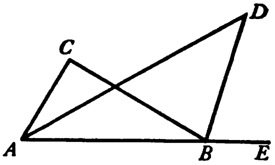
13. Given two angles of a triangle, with compasses and straightedge construct the third angle. Base the construction on § 48.

14. In this figure $\angle a$ equals the sum of what two angles? $\angle b$? $\angle c$? From these equations find the number of degrees in $\angle a + \angle b + \angle c$.



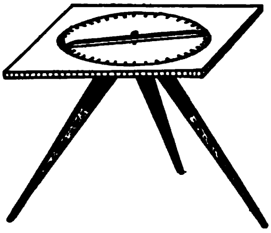
15. Prove that if AD bisects $\angle A$ of $\triangle ABC$, and BD bisects $\angle EBC$, the exterior angle, then $\angle D = \frac{1}{2} \angle C$.

SUGGESTION. — Express the sum of the angles of $\triangle ABC$ in terms of $\angle D, \angle A, \angle B$, and $\angle C$. Set this equal to $\angle A + \angle B + \angle C$. Why?

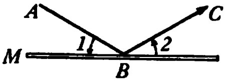


16. I may test my accuracy in measuring angles with a surveyor's instrument, as follows: I drive stakes in the ground at three points A, B, and C, so that they are not in a straight line. I set the instrument over A and measure $\angle BAC$. Then I set it over B and measure $\angle ABC$. Then I set it over C and measure $\angle ACB$. The angles are found to be 46° 12′, 32° 25′, and 101° 33′. Am I accurate?

NOTE. — It would be found practicable and interesting to make an instrument for measuring angles, such as that shown in the drawing, and to use it out of doors in measurements like those suggested in the above problem. The instrument would prove useful for many other out-of-door problems. It consists of a board (a drawing board is convenient) mounted upon a tripod. On the board is a circle marked off into 360°. A pointer, which is used for sighting to objects as well as for turning off angles, swings on a pivot at the center of the circle.

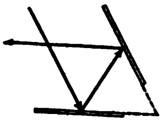


17. It is a law of science that when a ray of light strikes a plane mirror it is reflected in such a direction that the striking ray and reflected ray make equal angles with the mirror. Thus, if M is an edge view of the mirror and a ray AB is reflected in the direction BC, $\angle 1 = \angle 2$.

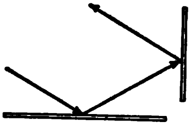


If a ray of light strikes first one mirror and then another such that the path of the ray forms a triangle with all of its angles equal, what is the angle between the mirrors?

SUGGESTIONS. — How many degrees are there in each angle of the triangle? Then how many degrees are there in each angle which the ray makes with each mirror?



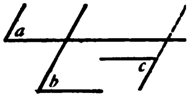
18. If a ray of light is reflected first by one mirror and then by another such that it leaves the second mirror along a line parallel to its path before it struck the first one, what is the angle between the mirrors?



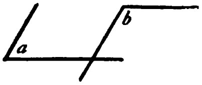
SUGGESTION. — At each point of reflection there are two equal angles and the sum of the three angles is 180°. Apply § 32 and § 48. Letter the angles and form equations. Solve for the angle between the mirrors.

19. Angles having their sides parallel, each to each, and extending in the same direction, or in opposite directions, from the vertices, are equal.

SUGGESTION. — Prove $\angle b = \angle a$ and $\angle c = \angle a$. Extend one side of either of the two angles to be proved equal, if necessary, until it meets the nonparallel side of the other angle. Show other positions in which the angles might be given, and see if the same proof applies.

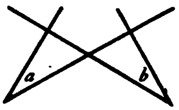


20. Angles having their sides parallel, each to each, one pair of parallel sides extending in the same direction, and the other pair in opposite directions, from the vertices, are supplementary.



SUGGESTION. — Prove $\angle a$ and $\angle b$ supplementary. Show other positions in which the angles might be given, and see if the same proof applies.

21. Two acute angles which have their sides perpendicular, each to each, are equal.

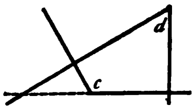


SUGGESTION. — Prove $\angle a = \angle b$. Use §48 and vertical angles.

22. Two obtuse angles which have their sides perpendicular, each to each, are equal.

SUGGESTION. — Extend one side of each angle through the vertex, thus forming two acute angles. What is the relation between the acute angles? What is the relation between each obtuse angle and the adjacent acute angle?

23. An acute angle and an obtuse angle which have their sides perpendicular, each to each, are supplementary.

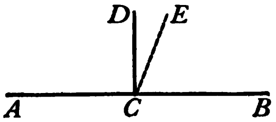


SUGGESTION. — Prove $\angle c$ and $\angle d$ supplementary. Extend a side of one of the angles through the vertex, forming an angle adjacent to that angle, and seek to use § 48 or a corollary of it. What is the relation of $\angle c$ to the adjacent angle? What is the relation of $\angle d$ to the same angle?

52. Indirect proof. — In the proofs of the preceding theorems, corollaries, and exercises, the method of reasoning consists of putting together known truths (definitions, axioms, theorems previously proved, the hypothesis, etc.), thus deriving new facts step by step, until the conclusion is reached. This is known as the **direct method of proof.** Another method of reasoning which may be used to advantage when the direct method is not convenient is called the **indirect method of proof.** It consists of *supposing that the conclusion which is to be established is not true, and proving that the supposition leads to a contradiction of a known truth, or an absurdity.* For this reason the indirect method of proof has long been called by the Latin term, **reductio ad absurdum** (meaning *reduction to an absurdity*).

The indirect method is illustrated in each of the following proofs.

53. Theorem. — *At a given point on a given line only one perpendicular can be drawn to the line.*



Hypothesis. C is a point on AB, and $CD \perp AB$.

Conclusion. CD is the only perpendicular to AB that can be drawn at C.

Proof. 1. Suppose that CD is not the only line that can be drawn perpendicular to AB at C, and that CE is a second perpendicular to AB.

2. Then $\angle BCD$ and $\angle BCE$ must be right angles. Def. $\perp$

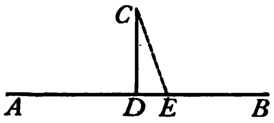
3. $\therefore \angle BCD$ must be equal to $\angle BCE$. § 20, II

4. But this is impossible. Ax. X

5. $\therefore$ the supposition is false, and CD is the only perpendicular that can be drawn to AB at C.

Draw a figure and write out the complete proof.

54. Theorem. — *Only one perpendicular to a given line can be drawn through a given point not on the line.*

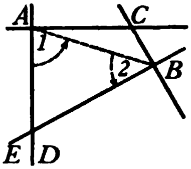


Hypothesis. C is a point not on AB, and $CD \perp AB$.

Conclusion. CD is the only perpendicular to AB that can be drawn through C.

Suggestions. Suppose that CD is not the only perpendicular to AB that can be drawn through C, and that CE is a second perpendicular. Then show that this supposition leads to a contradiction of § 50.

55. Theorem. — *Two straight lines perpendicular respectively to two intersecting straight lines must intersect.*



Hypothesis. AC and BC intersect at C, $AD \perp AC$ and $BE \perp BC$.

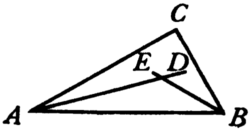
Conclusion. AD and BE intersect.

Proof. 1. Suppose that AD and BE do not intersect.

2. Then AD must be parallel to BE. § 9
3. Draw AB.
4. $\angle 1$ and $\angle 2$ must be supplementary. § 32
5. $\therefore \angle 1 + \angle 2$ must equal a st. $\angle$. Def. supp. $\angle$s
6. But $\angle DAC$ and $\angle CBE$ are right angles. Def. $\perp$
7. $\therefore \angle DAC + \angle CBE =$ st. $\angle$. § 17
8. $\therefore \angle 1 + \angle 2$ must equal $\angle DAC + \angle CBE$. Ax. I
9. But this is impossible. Ax. X
10. $\therefore$ the supposition is false, and AD and BE do intersect.

Write the proof in full.

56. Theorem. — *Lines drawn from the vertices of two angles of a triangle and lying within those angles must intersect.*



Hypothesis. In $\triangle ABC$, AD lies within $\angle A$ and BE lies within $\angle B$.

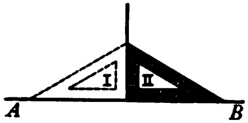
Conclusion. *AD* and *BE* intersect.

Suggestions. Suppose that *AD* and *BE* do not intersect. Then they must be parallel. Why? Hence, $\angle BAD + \angle EBA$ must equal a straight angle. Why? Now show that this is impossible by § 48.

Write the proof in full.

EXERCISES

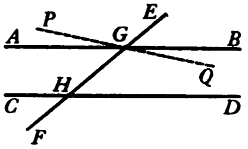
1. Test the accuracy of the right angle of your triangle by drawing a perpendicular to a line *AB*, using position I as in the figure, then turning the triangle over to position II to see if the perpendicular coincides with the edge of the triangle in its new position. Upon what principle does this test depend?



2. Prove the theorem in § 54 by using § 41.

3. Prove the theorem in § 36 by the indirect method.

SUGGESTION. — Suppose that *AB* is not parallel to *CD*, and that some other line *PQ* drawn through *G* is parallel to *CD*. Show that $\angle BGE$ would equal $\angle QGE$. Then show that this is impossible.



4–8. Prove the theorems in §§ 37, 38, 39, 40, and 41 by the indirect method.

9. In $\triangle ABC$, *AD* bisects $\angle A$, and *BE* bisects $\angle B$. Prove that *AD* and *BE* intersect.

CHAPTER III

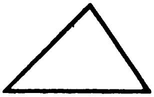
TRIANGLES: PROOF BY SUPERPOSITION

57. **Triangles classified as to sides.** — Triangles are classified according to the lengths of their sides, as follows:

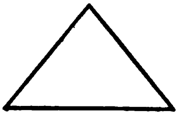
A **scalene** triangle is one that has *no two* sides equal.

An **isosceles** triangle is one that has *two* sides equal.

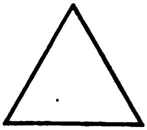
An **equilateral** triangle is one that has *all three* sides equal.



SCALENE



ISOSCELES



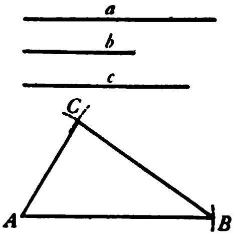
EQUILATERAL

In an *isosceles* triangle the vertex formed by the intersection of the two equal sides is called the **vertex** of the triangle, and the side opposite that vertex is called the **base**. Point out the vertex and the base of the above isosceles triangle.

EXERCISES

1. A triangle may be constructed with sides equal to three given line-segments, a, b, and c, as follows: Draw a line-segment AB equal to a. With center at A and radius equal to b, draw an arc. With center at B and radius equal to c, draw a second arc, meeting the first arc at a point C. Draw AC and BC. Then $\triangle ABC$ has its sides equal to a, b, and c by the construction.

Draw any three line-segments, then construct a triangle with sides equal to these three segments.



2. Construct an isosceles triangle with base equal to a given line-segment x and each of the equal sides equal to a given line-segment y.

3. Construct an isosceles triangle each of whose equal sides is twice as long as the base.

4. Construct an equilateral triangle with each side equal to a given line-segment *m*.

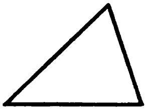
58. **Triangles classified as to angles.** — Triangles are classified according to their angles, as follows:

An **acute** triangle is a triangle all of whose angles are acute.

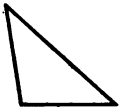
An **obtuse** triangle is a triangle one of whose angles is obtuse.

A **right** triangle is a triangle one of whose angles is a right angle.

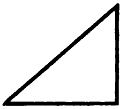
An **equiangular** triangle is a triangle all of whose angles are equal.



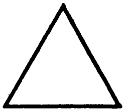
Acute



Obtuse



Right

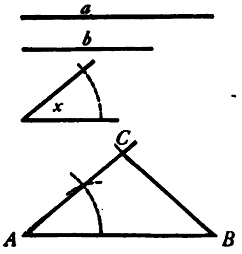


Equiangular

In a *right triangle* the side opposite the right angle is called the **hypotenuse**, and the other two sides are called the **legs**. Point out the hypotenuse of the above right triangle.

EXERCISES

1. A triangle may be constructed with two sides and the included angle (angle formed by the two sides) equal to two given line-segments *a* and *b* and the given angle *x*, respectively, as follows: Draw a line-segment AB equal to *a*. Construct $\angle BAC$ equal to $\angle x$ as in § 14. Mark off AC equal to *b*. Draw BC. Then $\triangle ABC$ has two sides and the included angle equal to *a*, *b*, and $\angle x$, respectively, by construction.



Draw any two line-segments and any angle, then construct a triangle with two sides and the included angle equal to these line-segments and angle, respectively.

2. Construct a right triangle with the legs equal to two given line-segments m and n. For the method of drawing the right angle, see the construction in § 23.

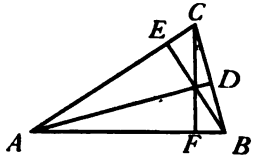
3. Construct an isosceles right triangle with each leg equal to a given line-segment s.

4. Construct a right triangle with one leg and the hypotenuse equal to two given line-segments a and b, respectively. Note that b must be longer than a.

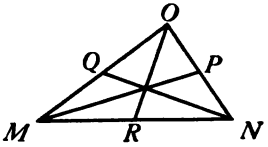
5. Construct a right triangle with the hypotenuse twice as long as one of the legs.

6. Construct a triangle with two angles and the included side (the side connecting the vertices of those angles) equal to two given angles x and y and a given line-segment a, respectively.

59. Altitudes and medians of a triangle. — A perpendicular to any side of a triangle from the opposite vertex is called an **altitude** of the triangle. How many altitudes has a triangle? Name the altitudes in △ *ABC*, and tell how each is drawn.



A line-segment joining any vertex of a triangle to the middle point of the opposite side is called a **median** of the triangle. How many medians has a triangle? Name the medians in △ *MNO*, and tell how each is drawn.



EXERCISES

1. Draw any triangle *ABC*, then draw the altitude from the vertex *C* to the side *AB*. See § 24 for the construction.

2. Draw an obtuse triangle. Draw the altitude from the vertex of one of the acute angles. Where does it meet the opposite side?

3. Draw a right triangle. Draw the altitude from the vertex of one of the acute angles. Where does it fall?

4. If an altitude of a triangle meets the opposite side produced through a vertex, what kind of triangle is it?

5. If an altitude of a triangle coincides with one of the sides, what kind of triangle is it?

6. If all altitudes of a triangle lie within the triangle, what kind of triangle is it?

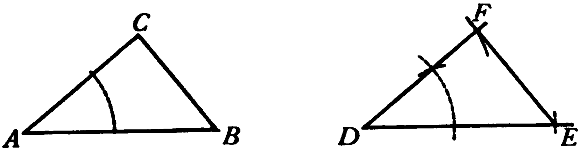
7. By using the construction in § 11, draw a median of a triangle.

60. Congruence of figures. — If one of two figures may be placed upon the other so that they coincide throughout, the figures are called **congruent.**

It is evident that if two figures are congruent, *for each part of one there is an equal part of the other.* The *equal parts* of two congruent figures are called **corresponding parts.**

In written work the words "is congruent to" are usually expressed by the symbol $\cong$. Thus the fact that $\triangle ABC$ is congruent to $\triangle DEF$ is written $\triangle ABC \cong \triangle DEF$.

61. Testing congruence of triangles. — Draw on paper (cardboard would be better if it were at hand) a large triangle



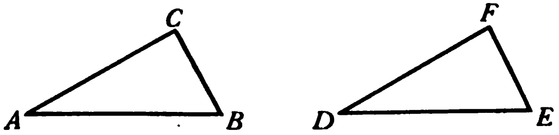
ABC. Then construct accurately a second triangle DEF so that $DE = AB$, $DF = AC$, and $\angle D = \angle A$. The method of construction is indicated in the above diagram. Explain it.

Having constructed $\triangle DEF$, cut it out with shears by cutting along the three sides. Now place $\triangle DEF$ upon $\triangle ABC$ so that DE coincides with its equal AB, D falling at A and E at B. When this is done, why does DF fall along AC? Since DF falls along AC, where does the vertex F fall? Why must it? Where does the side EF fall? Why must it? It is thus seen that the triangles are congruent, because they coincide.

62. Proof by superposition. — Instead of testing the congruence of figures by cutting one out and applying it to the other to see if they can be made to coincide, as in § 61, *the congruence may be proved by merely imagining one figure placed upon the other and reasoning out that they must coincide.* The imaginary placing of one figure upon the other is called **superposition.**

The following theorems show how triangles are proved congruent by the **method of superposition.**

63. Theorem. — *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $AC = DF$, and $\angle A = \angle D$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Proof. 1. $AB = DE$. Hyp.

2. $\therefore \triangle ABC$ may be placed upon $\triangle DEF$ so that AB coincides with DE, A falling upon D and B upon E. § 10

3. $\angle A = \angle D$. Hyp.

4. $\therefore AC$ must fall along DF. § 13

5. $AC = DF$. Hyp.

6. $\therefore C$ must fall upon F. § 10

7. $\therefore CB$ must coincide with FE. § 9, II

8. $\therefore \triangle ABC \cong \triangle DEF$. Def. Congruence

Draw two triangles accurately with the parts equal as given in the hypothesis, and write out the complete proof without reference to the book.

64. Corollary. — *Two right triangles which have the legs of one equal respectively to the legs of the other are congruent.*

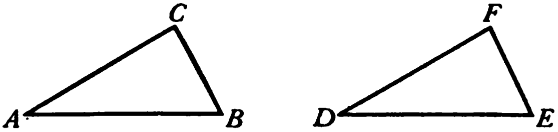
The proof is left to the student.

EXERCISES

1. Prove orally the theorem of § 63 by showing first that $\triangle ABC$ may be placed upon $\triangle DEF$ so that AC coincides with DF, A falling upon D and C upon F.

2. Prove orally the theorem of § 63 by showing first that $\triangle ABC$ may be placed upon $\triangle DEF$ so that $\angle A$ coincides with $\angle D$.

65. Theorem. — *If two angles and the included side of one triangle are equal respectively to two angles and the included side of another, the triangles are congruent.*



NOTE. — If desirable, before attempting the following proof, triangles may be constructed on paper or cardboard with two angles and the included side of one equal respectively to two angles and the included side of the other, and one of them cut out and applied to the other as in § 61.

Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$, and $AB = DE$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Suggestions. Since $\triangle ABC$ may be placed upon $\triangle DEF$ so that one pair of equal parts coincide, in how many ways may this be done? If AB is made to coincide with its equal DE, A falling upon D and B upon E, where must AC fall? Why? At the same time where must BC fall? Why? Why, then, at the same time must C fall upon F?

Write out a complete proof.

66. Corollary 1. — *If two angles and any side of one triangle are equal respectively to two angles and the similarly placed side of the other, the triangles are congruent.*

For, by § 49, the third pair of angles must be equal, and hence the triangles must be congruent by § 65.

67. Corollary 2. — *Two right triangles are congruent if a leg and an acute angle of one are equal respectively to a leg and an acute angle of the other.*

The proof is left to the student.

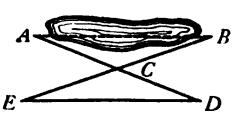
68. Corollary 3. — *Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*

The proof is left to the student.

69. Distances and angles compared by congruence of triangles. — Since the corresponding parts of congruent triangles are equal, angles or distances may be proved equal by proving that triangles of which they are corresponding parts are congruent. This is one of the chief uses of congruent triangles, and is illustrated in the following exercises.

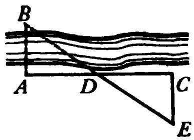
EXERCISES

1. In order to find the distance from A to B across a lake, surveyors measured off a straight line AC and extended it to D so that $CD = AC$. Then they measured the distance BC and extended BC to E so that $CE = BC$. Then they measured the distance from E to D. Prove that AB equals the measured distance DE.

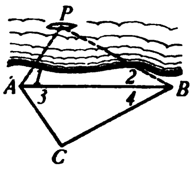


SUGGESTION: — Prove the triangles congruent.

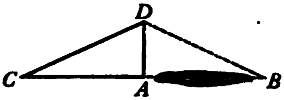
2. The distance AB across a stream may be found as follows: Measure AC at right angles to AB. Locate a point D halfway between A and C. Measure CE at right angles to AC, to a point E in line with B and D. Prove that AB equals CE.



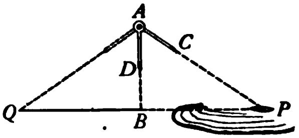
3. The Greeks, as early as the time of Thales (640–546 B.C.), determined the distance AP of a ship from shore by means of the congruence of triangles having two angles and the included side of one equal respectively to two angles and the included side of the other. They measured $\angle 1$ and $\angle 2$. Explain how they were then able to mark off a distance on the shore equal to AP and thus find the distance AP.



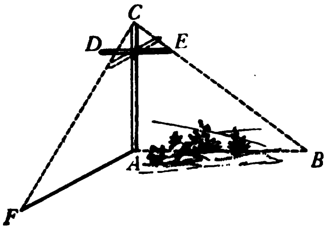
4. The distance from A to an inaccessible point B may be found approximately without instruments as follows: Stand at A and look at B, raising or lowering the head until your hatbrim falls in the line of sight. Then, without raising or lowering the head, turn about, and observe the point C at which the line of sight strikes the ground. Pace the distance from A to C. In the diagram, D represents the position of the eyes. Prove that $AB = AC$.



5. Thales (See Ex. 3) is said to have made an instrument for determining the distance BP of a ship from shore. It consisted of two rods AC and AD, hinged together at A. Rod AD was held vertically over point B, while rod AC was pointed toward P. Then, without changing $\angle DAC$, the instrument was revolved about AD, and point Q noted on the ground at which the arm AC was directed. BQ was then measured. Prove that $BP = BQ$.

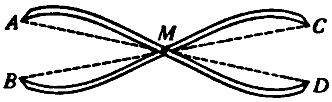


6. An instrument used as late as the sixteenth century for finding the distance from A to an inaccessible point B was called a *cross-staff.* It consisted of a vertical staff AC to which was attached a horizontal cross-bar DE that could be moved up or down on the staff. Sighting from C to B, DE was lowered or raised until C, E, and B were in a straight line. Then the instrument was revolved about CA and the point F at which the line of sight CE met the ground marked. The distance AF was then measured. Prove that $AB = AF$.

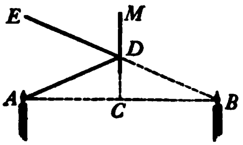


NOTE. — Students may easily make and use this instrument.

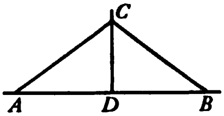
7. The calipers shown in the drawing may be used for measuring the thickness of objects, such as the diameters of pipes, the diameters of trees in forestry, etc. AD and BC are pivoted at M. A, M, and D are in a straight line, and B, M, and C, in a straight line. M is the middle point of both AD and BC. Prove that the distance between A and B equals the distance between C and D. Hence explain how the instrument may be used.



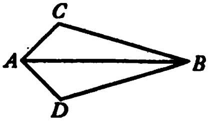
8. Every one is familiar with the fact that if an object is placed before a plane mirror, its image appears to be as far behind the mirror as the object is in front. M is an edge view of a mirror. Light from an object at A strikes the mirror at D and is reflected to the eye at E. The mind projects the line ED through the mirror to B, forming the image at B. It is known from science that $\angle MDE = \angle ADC$ and that $CD \perp AB$. Prove that $CB = AC$.



9. If two oblique line-segments drawn from a point on a perpendicular to a line cut off equal distances on the line from the foot of the perpendicular, prove that the oblique segments are equal.

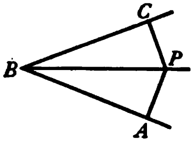


10. In the adjoining figure, AB bisects $\angle A$ and also $\angle B$. Prove that $\angle C = \angle D$.



11. Prove that if the bisector of the angle at a vertex of a triangle is perpendicular to the opposite side, the triangle must be isosceles.

12. In the annexed figure, BP is the bisector of $\angle ABC$, and $PC \perp BC$ and $PA \perp BA$. Prove that $PC = PA$.



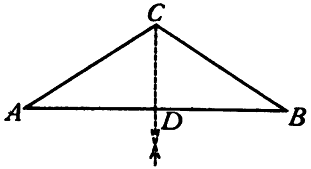
13. Prove that the medians drawn to the equal sides of an isosceles triangle are equal.

14. Prove that all of the medians of an equilateral triangle are equal.

15. Prove that the altitudes to the equal sides of an isosceles triangle are equal.

16. Prove that all of the altitudes of an equilateral triangle are equal.

70. Theorem. — *In any isosceles triangle the angles opposite the equal sides are equal.*



Hypothesis. $\triangle ABC$ is isosceles, with $AC = BC$.

Conclusion. $\angle A = \angle B$.

Suggestions. It may be shown that $\angle A = \angle B$ by showing that they are corresponding angles of congruent triangles. How must CD be drawn in order to prove that $\triangle ACD \cong \triangle BCD$ by means of § 63? Hence begin by drawing the bisector CD of $\angle C$. Draw an accurate figure with straightedge and compasses, and write the complete proof.

71. Corollary 1. — *An equilateral triangle is also equiangular.*
The proof is left to the student.

72. Corollary 2. — *Each angle of an equilateral triangle is* 60°.
The proof is left to the student.

73. Corollary 3. — *In any isosceles triangle the bisector of the angle at the vertex, the median to the base, the altitude to the base, and the perpendicular bisector of the base all coincide.*

Suggestion. In the proof of § 70 it was proved that CD, the bisector of the angle at the vertex, formed congruent triangles, $\triangle ACD$ and $\triangle BCD$. Show from this that CD is also the median to the base, altitude to the base, and perpendicular bisector of the base.

EXERCISES

1. Prove that if the angle at the vertex of an isosceles triangle is 60°, each of the other angles is 60°.

2. One of the angles at the base of an isosceles triangle is 45°. Find the size of each of the other angles.

3. Construct an angle of 60° by use of compasses and straightedge.

4. Construct an angle of 30° by use of compasses and straightedge.

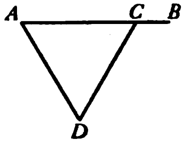
5. The angle at the vertex of an isosceles triangle is three times as large as either of the other angles. Find by an equation the number of degrees in each angle of the triangle.

6. The angle at the vertex of an isosceles triangle is 20° more than the sum of the other two angles. Find by an equation the number of degrees in each angle.

7. One of the angles at the base of an isosceles triangle is 37° 35′. Find the size of each of the other angles.

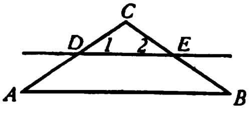
8. Trisect a right angle by use of compasses and straightedge.

9. A surveyor often needs to lay off an angle of 60° from a given line. If AB is the given line, and A the point at which the angle is to be laid off, prove that he may do this as follows: With one end of a 50-foot tape held at A, and the other at C, which is 25 ft. from A, let a third person take the tape at the 25-foot mark and stretch both parts straight, locating a point D. Then $\angle DAC = 60°$.

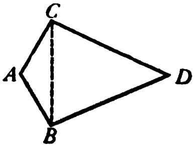


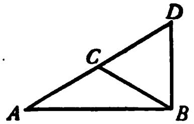
Lay off an angle of 60° in the school yard.

10. A straight line drawn through the equal sides of an isosceles triangle, and parallel to the base, makes equal angles with the equal sides. Prove $\angle 1 = \angle 2$.



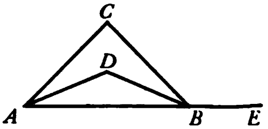
11. In the *kite* shown in the figure, the edges AC and AB are equal and CD and BD are equal. Prove that $\angle C = \angle B$.





12. If one of the equal sides of an isosceles triangle be produced through the vertex its own length, the straight line joining its extremity to the nearer extremity of the base is perpendicular to the base.

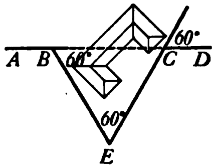
Suggestion. — Show that $\angle DBA$ = one half of the sum of the angles of $\triangle ABD$.



13. The angle made by the bisectors of the base angles of an isosceles triangle is equal to an exterior angle at the base. (Prove $\angle ADB = \angle EBC$.)

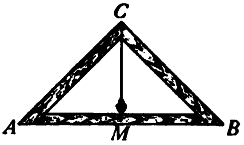
Suggestion. — Show that each angle is a supplement of $\angle ABC$.

14. In surveying a straight line AB, surveyors run into an obstacle. Prove that they may pass the obstacle and continue the line CD in line with AB as follows: Measure off an angle of 60° at B, and run the line BE sufficiently long to clear the obstacle. At E construct an angle of 60° as in the diagram, and measure off $EC = BE$. Then at C turn off an angle of 60°, and establish the line CD. CD is a prolongation of AB.

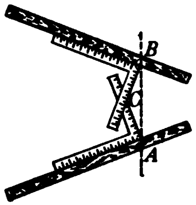


Note. — It would be found a valuable problem for the students actually to run a straight line through an obstacle, such as the schoolhouse, as suggested in the above exercise, by using the instrument described in the note, page 40, for turning off the angles.

15. An instrument for leveling, called a *plumb level*, used before the modern spirit or bubble level was invented, consisted of a frame in the form of an isosceles triangle with AC and BC as equal sides. The middle point M of the base was marked. A plumb line was suspended from a pivot at C. In using the instrument, it was held in an upright position and the edge AB rested upon the surface to be leveled. Prove that when the plumb line hung directly over the mark M the surface upon which the edge AB rested was level. (*Level* is at right angles to *vertical*.)

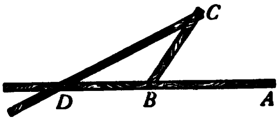


Note. — Students will find it easy and interesting to make such an instrument of wood. The plumb level is an accurate and quickly constructed substitute for use in any practical work where a spirit level is not at hand.



16. For marking a line that will cut two converging boards at equal angles, a carpenter places two steel squares against the boards, as shown in the drawing, and adjusts them so that $AC = BC$. Prove that the line AB then makes equal angles with the edges of the two boards.

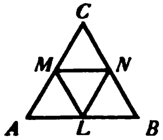
17. An instrument for bisecting an angle consists of three bars AB, BC, and CD, pivoted together at B, C, and D. The pivot D works in a groove of the bar CD so that by lengthening or shortening CD it is possible to revolve BC about the pivot B and reduce or enlarge $\angle ABC$. BC and BD are equal. Prove that when $\angle ABC$ is adjusted to coincide with any given angle, $\angle BDC$ is equal to one half of that angle.



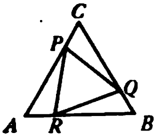
18. Prove that the bisectors of the base angles of an isosceles triangle are equal.

19. Prove that all of the bisectors of the angles of an equilateral triangle are equal.

20. $\triangle ABC$ is an equilateral triangle, and points M, N, and L are the middle points of its sides. Prove that $\triangle MNL$ is equilateral.

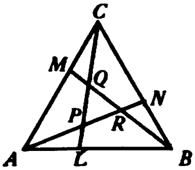


21. $\triangle ABC$ is an equilateral triangle. Points P, Q, and R are taken on sides AC, BC, and AB, respectively, so that $AP = CQ = BR$. Prove that $\triangle PQR$ is equilateral.

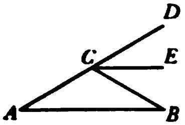


22. $\triangle ABC$ is equilateral, and $AM = CN = BL$. AN, BM, and CL intersect at P, Q, and R, forming $\triangle PQR$, as in the figure. Prove $\triangle PQR$ equilateral.

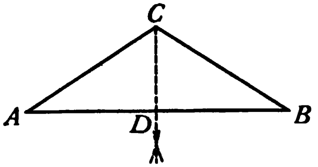
SUGGESTION. — $\triangle PQR$ may be proved equilateral if it may be shown that $PL = RN = QM$, $AP = BR = CQ$, and $AN = BM = CL$. These equalities may be proved by congruent triangles.



23. Prove that the bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.



74. Theorem. —*If two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*



Hypothesis. In $\triangle ABC$, $\angle A = \angle B$.

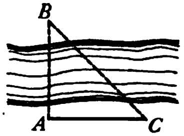
Conclusion. $AC = BC$, and $\triangle ABC$ is isosceles.

Suggestions. How may CD be drawn in order that $\triangle ACD$ and $\triangle BCD$ may be proved congruent? Write the complete proof.

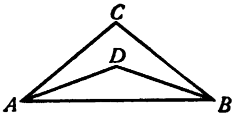
75. Corollary. —*An equiangular triangle is also equilateral.*

The proof is left to the student.

EXERCISES

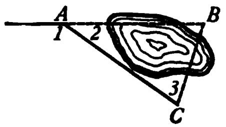


1. In order to determine the distance from A to the inaccessible point B, a line AC is measured off at right angles to AB, and a point C found in this line at which $\angle ACB = 45°$. Prove that $AB = AC$.



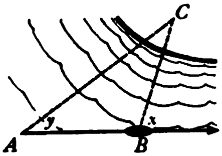
2. If $\triangle ABC$ is isosceles, with $AC = BC$, and AD and BD bisect $\angle A$ and $\angle B$, respectively, prove that $\triangle ADB$ is isosceles.

3. If the angle at the vertex of an isosceles triangle is equal to one half of a base angle, the bisector of a base angle divides the triangle into two isosceles triangles.



4. In surveying, the distance from A to the inaccessible point B may be obtained as follows: Measure off AC until a point is found from which $\angle 3 = \frac{1}{2}\angle 1$. Then $AB = AC$. Prove it.

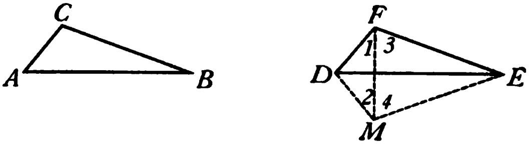
5. The distance BC from his ship to an object such as a lighthouse C may be determined by a sailor as follows: When the ship is at any point A he observes $\angle y$, the angle which the object C makes with the course AB of the ship. Then he observes when the object makes an angle just twice as large with the course of the ship, *i.e.* when $\angle x = 2\angle y$. From the ship's log he knows how far the ship has traveled from A to B. Prove that this equals the distance from B to C.



6. If one acute angle of a right triangle is 30° and the other one 60°, the hypotenuse is twice as long as the shorter leg.

Suggestion. — Extend the shorter leg its own length through the vertex of the right angle, and connect its extremity with the other extremity of the hypotenuse. Prove the triangle thus formed isosceles.

76. Theorem. — *If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $AC = DF$, and $BC = EF$.

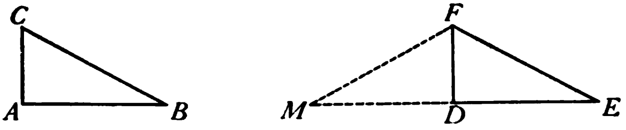
Conclusion. $\triangle ABC \cong \triangle DEF$.

Suggestions. If it were first proved that $\angle C = \angle F$, the triangles would be congruent by what theorem? For proving that $\angle C = \angle F$, show that $\triangle ABC$ may be placed along side of $\triangle DEF$, so that A falls at D, B at E, and C at M, on the opposite side of DE from F. Then it may be proved that $\angle F = \angle M$ by drawing FM and proving what angles equal? How does the proof that $\angle F = \angle M$ when FM does not intersect DE between D and E differ from that when it does?

Hence begin by showing that $\triangle ABC$ may be placed in the position of $\triangle DME$ and drawing FM. Write out the complete proof.

Note. — Observe that the method of superposition which was used in § 63 and § 65 cannot be used in the theorem above, because, since no relations of angles are known, if $\triangle ABC$ were placed upon $\triangle DEF$ so that AB coincided with its equal DE, it would be impossible to prove where AC and BC must fall. A proof similar to that used above will be used also in the following theorem.

77: Theorem. — *If the hypotenuse and a leg of one right triangle are equal respectively to the hypotenuse and a leg of another, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\angle A =$ rt. $\angle$, $\angle D =$ rt. $\angle$, $BC = EF$ and $AC = DF$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Suggestions. If it is first proved that $\angle B = \angle E$, how may the triangles be proved congruent? For proving that $\angle B = \angle E$, show that $\triangle ABC$ may be placed along side of $\triangle DEF$ (turning it face downward if necessary), so that A falls at D, C at F, and B at M, on the opposite side of FD from E. If the line MDE is then proved to be a straight line, the figure MFE is what kind of figure?

Hence, begin by showing that $\triangle ABC$ may be placed in the position of $\triangle DMF$, and proving that MFE is an isosceles triangle. Write the proof in full.

78. Summary of theorems on congruent triangles. — It has been shown in this chapter that *triangles in general* are congruent under *three* conditions, namely, when they have the following like parts equal:

(1) *Two sides and the included angle.*
(2) *Two angles and the included side (or any side).*
(3) *Three sides.*

It has been shown that two *right triangles* are congruent under *four* conditions, namely, when they have the following like parts equal:

(1) *Two legs.*
(2) *A leg and an acute angle.*
(3) *The hypotenuse and an acute angle.*
(4) *The hypotenuse and a leg.*

The student should master these seven principles now, since they will be much used in the more advanced chapters of geometry.

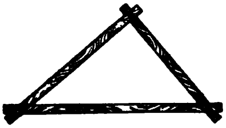
EXERCISES

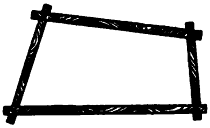
1. Draw any triangle. Then, with compasses and straightedge, construct a second triangle with sides equal to the sides of the first. Cut out the second triangle and apply it to the first, and thus verify the theorem in § 76.

2. Nail three strips of wood together so as to form a triangle, using only one nail at each joint. Is this frame rigid, or can it be changed into different shapes by exerting pressure upon it?

In the same way, nail four strips of wood together as shown in the drawing. Can this frame be changed into different shapes by exerting pressure upon it?

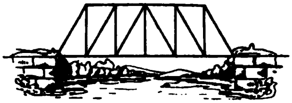
Show that the fact that a triangular frame is rigidly fixed when its three sides are of fixed lengths follows from § 76.



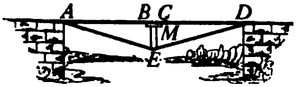


3. Why is a roof sufficiently braced when a board is nailed across each pair of rafters?

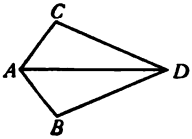
4. Why is a long span of a bridge in which the truss is made with queen posts and diagonal rods, as shown in the drawing, sufficiently supported?



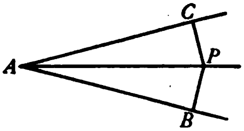
5. The drawing shows a piece of a bridge, called an inverted kingpost truss. AB and CD are steel bars resting on a plate M. The plate M is supported by wire ropes EA and ED. Show why this structure will support a heavy weight placed upon it.



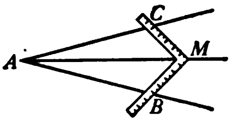
6. In the *kite* shown in the figure, $AB = AC$ and $BD = CD$. Prove that AD bisects $\angle BAC$ and $\angle BDC$.



7. In the angle BAC, $PC \perp AC$, $PB \perp AB$, and $PC = PB$. Prove that AP bisects $\angle BAC$.



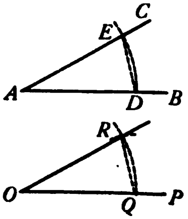
8. A carpenter bisects an angle BAC with his steel square as follows: He marks off AC and AB of equal length on the sides of the angle. Then he places his square so that $MC = MB$, as shown in the drawing. Then he marks the point M at the heel of the square, and draws AM. Prove that AM bisects $\angle BAC$.



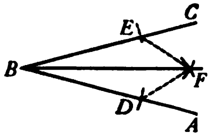
NOTE. — The following important constructions were assumed in Chapter I to be accurate. It is essential that the accuracy of these constructions now be proved.

9. Draw any angle BAC, then construct a second angle POR equal to it as in § 14. Prove that the construction is accurate.

SUGGESTION. — Draw ED and QR, and prove $\triangle EAD \cong \triangle ROQ$.

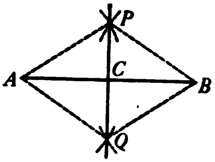


10. Bisect a given angle *ABC* as in § 16. Prove that the construction is accurate.



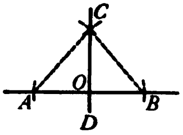
Suggestion. — Draw *EF* and *DF*, and prove $\triangle EBF \cong \triangle DBF$.

11. Bisect a given line-segment *AB* as in § 11. Prove that the construction is accurate.

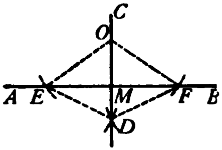


Suggestion. — Draw *AP*, *BP*, *AQ*, and *BQ*. For proving that $AC = CB$ what triangles must be proved congruent? For proving these triangles congruent what angles must first be proved equal? For proving these angles equal begin by proving $\triangle APQ \cong \triangle BPQ$.

12. Draw a perpendicular to a given line *AB* at a given point *O* as in § 23. Prove that the construction is accurate.



13. Draw a perpendicular to a given line *AB* from a given external point *O* as in § 24. Prove that the construction is accurate.



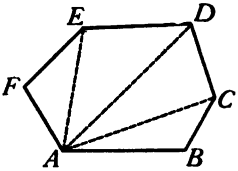
Suggestion. — For proving $CD \perp AB$ it is necessary first to prove $\triangle EMO \cong \triangle FMO$. Why? For proving $\triangle EMO \cong \triangle FMO$ it is necessary first to prove what angles equal? For proving these angles equal it is necessary to begin by proving what triangles congruent?

CHAPTER IV

POLYGONS

79. Polygons. — The plane figure formed by three or more line-segments that completely enclose part of a plane is called a **polygon**.

The line-segments are the **sides**, and the points where they meet are the **vertices**, of the polygon. The angles formed by the adjacent sides, as ∠ *ABC*, ∠ *BCD*, etc., in the figure, are the **interior angles**, or merely the **angles**, of the polygon. What relation exists between the number of sides, the number of vertices, and the number of angles of a polygon?



A polygon is named by naming its vertices, taken in order. Thus, the figure above is named *ABCDEF*.

A polygon no side of which produced will enter the surface enclosed is **convex**. Otherwise it is **concave**. Draw a concave polygon.

A polygon is **equilateral** if all of its sides are equal, and **equiangular** if all of its angles are equal.

A polygon is **regular** if it is both equilateral and equiangular.

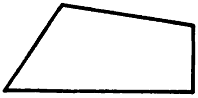
A line-segment connecting two non-adjacent vertices of a polygon is called a **diagonal**. Thus, in the figure, *AC*, *AD*, and *AE* are diagonals.

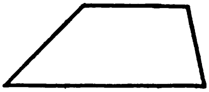
The sum of the sides of a polygon is called its **perimeter**.

80. Polygons classified as to the number of sides. — Polygons are classified according to the number of sides. A polygon of three sides is a **triangle**. A polygon of four sides is called a **quadrilateral**; one of five sides, a **pentagon**; one of six sides, a **hexagon**; one of eight sides, an **octagon**; one of ten sides, a **decagon**; etc.

81. Quadrilaterals. — A **trapezium** is a quadrilateral having no sides parallel.

A **trapezoid** is a quadrilateral having one and only one pair of opposite sides parallel. The parallel sides of a trapezoid are called the **bases**. The perpendicular distance between the bases is called the **altitude**. An **isosceles** trapezoid is one of which the non-parallel sides are equal.

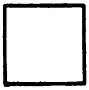

Trapezium

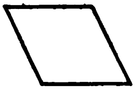

Trapezoid

A **parallelogram** is a quadrilateral having both pairs of opposite sides parallel. Any one of the four sides of a parallelogram may be considered the **base**, and the perpendicular distance between it and the opposite side, the **altitude**.

Parallelogram

Rectangle


Square

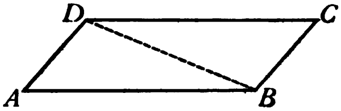

Rhombus

A **rectangle** is a parallelogram all of whose angles are right angles.

A **square** is a rectangle all of whose sides are equal.

A **rhombus** is a parallelogram all of whose sides are equal but whose angles are not right angles.

82. Theorem. — *The opposite sides of a parallelogram are equal.*



Hypothesis. $ABCD$ is a parallelogram.

Conclusion. $AB = DC$ and $AD = BC$.

Proof. 1. $ABCD$ is a parallelogram. Hyp.
2. $\therefore AB \parallel DC$ and $AD \parallel BC$. Def. ▱
3. Draw diagonal DB.
4. Then in $\triangle ABD$ and $\triangle BCD$, DB is a common side.
5. $\angle ABD = \angle BDC$ and $\angle ADB = \angle DBC$. § 29
6. $\therefore \triangle ABD \cong \triangle BCD$. § 65
7. $\therefore AB = DC$ and $AD = BC$. Def. Congruence

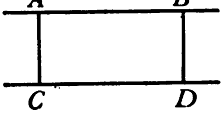
83. Corollary 1. — *A diagonal divides a parallelogram into two congruent triangles.*

This follows from step 6 in § 82.

84. Corollary 2. — *Segments of parallel lines cut off by two parallel lines are equal.*

The proof is left to the student.

85. Corollary 3. — *Two parallel lines are everywhere equidistant.*

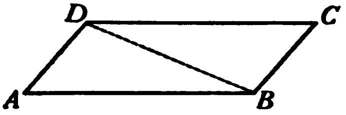


Suggestion. If $AB \parallel CD$, they may be proved everywhere equidistant by proving what equal? If A and B are any two points of AB, and $AC \perp CD$ and $BD \perp CD$, prove that AC and BD are parallel and hence equal.

86. Theorem. — *The opposite angles of a parallelogram are equal, and the consecutive angles supplementary.*

The proof is left to the student. Write the proof in full.

87. Theorem. — *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

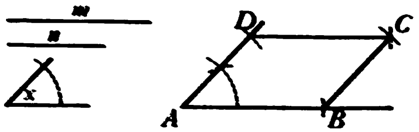


Hypothesis. In quadrilateral $ABCD$, $AB = DC$ and $AD = BC$.

Conclusion. $ABCD$ is a parallelogram.

Suggestions. Draw diagonal DB. It may be proved that $ABCD$ is a parallelogram by proving what lines parallel? For proving these lines parallel, what angles should be proved equal? How may these angles be proved equal? Hence begin by proving $\triangle ABD \cong \triangle BCD$. Write out the complete proof.

88. Construction. — *Construct a parallelogram which shall have two adjacent sides and the included angle equal respectively to two given line-segments and a given angle.*



Given line-segments m and n and $\angle x$.

Required to construct a parallelogram which shall have two adjacent sides and the included angle equal respectively to m, n, and $\angle x$.

Construction. 1. Construct $\angle BAD = \angle x$.

2. Mark off $AB = m$ and $AD = n$.

3. With center B and radius equal to n, draw an arc; and with center D and radius equal to m, draw an arc, intersecting the first arc at C.

4. Draw BC and DC.

5. Then $ABCD$ is the required parallelogram.

Proof. 1. $BC = AD$ and $DC = AB$. Construction

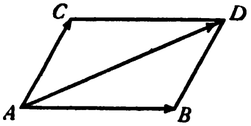
2. $\therefore ABCD$ is a parallelogram. § 87

3. $\angle BAD = \angle x$, $AB = m$, $AD = n$. Construction

4. Hence $ABCD$ is the required parallelogram.

EXERCISES

1. If two forces are exerted in different directions upon the same object at A, they have the same effect as a single force called their *resultant*. If the directions and magnitudes of the two forces are represented by the line-segments AB and AC, the direction and magnitude of the resultant will be represented by the line-segment AD, diagonal of the parallelogram $ABDC$.



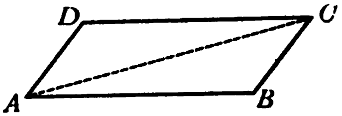
A force of 100 lb. and another of 200 lb. are exerted upon an object at an angle of 45° with each other. Representing 100 lb. by a line-segment 2 in. long, draw the forces to scale and find the resultant. (Use protractor. Measure diagonal with ruler and compute resultant in pounds.)

2. Two forces are exerted upon an object at right angles with each other. One force is 400 lb. and the other 600 lb. Construct and compute their resultant as in Ex. 1.

3. Two forces, one of 48 lb. and the other of 60 lb., are exerted upon an object at an angle of 120° with each other. Construct and compute their resultant.

4. When a train is approaching a station at a speed of 40 ft. per second, a mail bag is thrown from the car, at right angles to the railroad track, with a speed of 20 ft. per second. Find the horizontal direction and speed of the bag. (Construct as in case of forces.)

89. Theorem. — *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

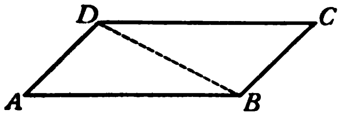


Hypothesis. In quadrilateral $ABCD$, $\angle A = \angle C$ and $\angle B = \angle D$.

Conclusion. $ABCD$ is a parallelogram.

Suggestions. It may be proved that $AB \parallel DC$ by first proving that $\angle B$ and $\angle C$ are supplementary. Draw AC. What is the sum of the angles of $\triangle ABC$ and $\triangle ADC$, or of $ABCD$? Since $\angle A = \angle C$ and $\angle B = \angle D$, show by equations that $\angle B + \angle C =$ st. $\angle$. Hence show that $AB \parallel DC$. By similar proof, $AD \parallel BC$. Write the proof in full.

90. Theorem. — *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

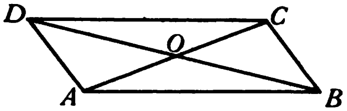


Hypothesis. $AB \parallel DC$ and $AB = DC$.

Conclusion. $ABCD$ is a parallelogram.

Suggestions. It may be proved that $ABCD$ is a parallelogram by proving that $AD \parallel BC$. For proving $AD \parallel BC$, what angles must first be proved equal? How may these angles be proved equal? Hence begin by drawing DB and proving the triangles congruent. Write out the complete proof.

91. Theorem. — *In any parallelogram the diagonals bisect each other.*



Hypothesis. In ▱ $ABCD$, AC and BD intersect at O.

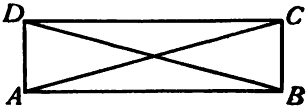
Conclusion. AC and BD bisect each other at O.

Suggestions. It may be proved that AC and BD bisect each other by proving what line-segments equal? These line-segments may be proved equal by proving what triangles congruent? Write out the complete proof.

92. Theorem. — *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

The proof is left to the student.

93. Theorem. — *In any rectangle the diagonals are equal.*



Hypothesis. $ABCD$ is a rectangle whose diagonals are AC and BD.

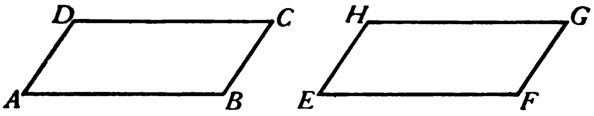
Conclusion. $AC = BD$.

Suggestions. Prove $\triangle ADB \cong \triangle ABC$. Write out the complete proof.

94. Theorem. — *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

The proof is left to the student.

95. Theorem. — *Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Hypothesis. *ABCD* and *EFGH* are parallelograms having $AB = EF$, $AD = EH$, and $\angle A = \angle E$.

Conclusion. ▱ *ABCD* $\cong$ ▱ *EFGH*.

Proof. 1. *ABCD* and *EFGH* are parallelograms having $AB = EF$, $AD = EH$, and $\angle A = \angle E$. Hyp.

2. $\therefore$ ▱ *ABCD* may be placed upon ▱ *EFGH* so that $\angle A$ coincides with $\angle E$, *AD* falling along *EH* and *AB* along *EF*. § 13

3. Then *D* will fall on *H* and *B* on *F*. § 10

4. Now *DC* and *HG* are both drawn through *H* and are parallel to *EF*. Def. ▱

5. $\therefore$ *DC* must fall along *HG*. § 46

6. Similarly, *BC* must fall along *FG*.

7. $\therefore$ *C* must fall on *G*. § 9, III

8. $\therefore$ ▱ *ABCD* $\cong$ ▱ *EFGH*. Def. Congruence

In what other ways might ▱ *ABCD* be placed upon ▱ *EFGH* in step 2?

96. Corollary. — *Two rectangles having equal bases and equal altitudes are congruent.*

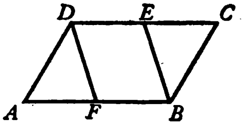
Two squares having a side of one equal to a side of the other are congruent.

The proof is left to the student.

EXERCISES

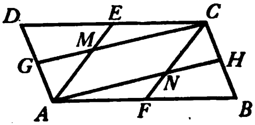
1. If one angle of a parallelogram is a right angle, all four angles are right angles and the figure is a rectangle.

2. If *ABCD* is a parallelogram, and *E* and *F* the middle points of *DC* and *AB*, respectively, then *BEDF* is also a parallelogram.



3. If *ABCD* is a parallelogram, *E*, *F*, *G*, and *H* the middle points of *DC*, *AB*, *AD*, and *BC*, respectively, and *AE* and *CG* intersect at *M* and *AH* and *CF* intersect at *N*, then *ANCM* is also a parallelogram.

4. The diagonals of a rhombus are perpendicular to each other.



5. The diagonals of a square bisect the angles of the square.

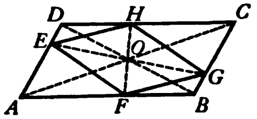
6. If the diagonals of a rectangle are perpendicular to each other, the figure is a square.

7. The perpendiculars to either diagonal of a parallelogram from the opposite vertices are equal.

8. The intersection of the diagonals of a parallelogram is called the *center* of the parallelogram. Prove that every line-segment drawn through the center of a parallelogram and terminating in the sides is bisected at the center.

9. If the vertices of one parallelogram lie upon the four sides of another, the parallelograms have the same center.

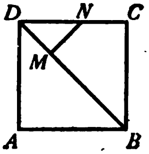
SUGGESTIONS. — Draw the diagonals *AC* and *BD*, intersecting at *O*. Draw *OE*, *OF*, *OG*, *OH*, and prove *EOG* and *FOH* straight lines and hence diagonals.



10. If from any point in the base of an isosceles triangle parallels to the sides are drawn, the parallelogram thus formed has the same perimeter for all positions of the point.

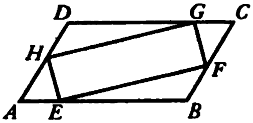
11. *ABCD* is a square. On the diagonal *BD*, *BM* is taken equal to *BC*, and *MN* is drawn perpendicular to *BD*. Prove that $DM = MN = NC$.

12. If either diagonal of a parallelogram bisects one of the angles, the figure is equilateral.



13. The bisectors of the angles of a parallelogram form a rectangle.

14. In the parallelogram $ABCD$, if points E, F, G, and H are so taken on the sides AB, BC, CD, and DA, respectively, that $AE = BF = CG = DH$, prove that $EFGH$ is a parallelogram.

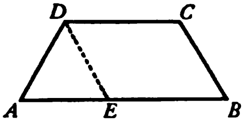


15. In the parallelogram $ABCD$, if E, F, G, and H are so taken on the sides AB, BC, CD, and DA, respectively, that $AE = AH = CF = CG$, prove that $EFGH$ is a parallelogram.

16. In the parallelogram $ABCD$, if E, F, G, and H are so taken on the sides AB, BC, CD, and DA, respectively, that $AE = CG$ and $BF = DH$, prove that $EFGH$ is a parallelogram.

17. In an isosceles trapezoid the non-parallel sides make equal angles with either of the parallel sides.

Suggestion. — Draw $DE \parallel CB$.

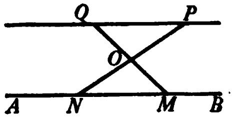


18. If the angles made by the non-parallel sides of a trapezoid with either of the parallel sides are equal, the trapezoid is isosceles.

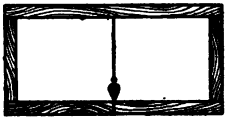
19. The diagonals of an isosceles trapezoid are equal.

20. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

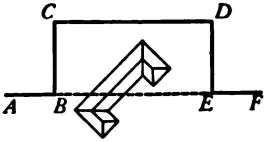
21. In order to establish a second point that with a given point P will determine a line parallel to a given line AB, a surveyor proceeds as follows: He measures off a line from P to a point N in AB, and locates O, the middle point of this line. Then from M, a second point in AB, he runs a line through O, and extends it to a point Q so that $OQ = MO$. Show that $QP \parallel AB$.



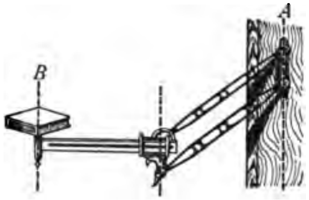
22. The figure shows a form of instrument sometimes used for leveling. It consists of a rectangular frame with a plumb line suspended from the middle point of the upper side. A mark is placed at the middle point of the lower side. In using the instrument to see if a surface is level, the lower side is placed on the surface to be leveled. Show that if then the plumb line hangs directly over the mark, the surface is level.



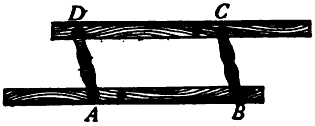
23. In surveying, a method of extending a line AB beyond an obstacle is as follows: A line BC is run perpendicular to AB, then CD is run perpendicular to BC, then DE is run perpendicular to CD and equal to BC, then EF is run perpendicular to DE. Prove that EF and AB lie in one straight line.



24. An adjustable bracket, as shown in the figure, is fastened to the wall at A, and carries a shelf B. Explain why the shelf B remains horizontal in all positions when the bracket is moved so that the shelf is raised and lowered.

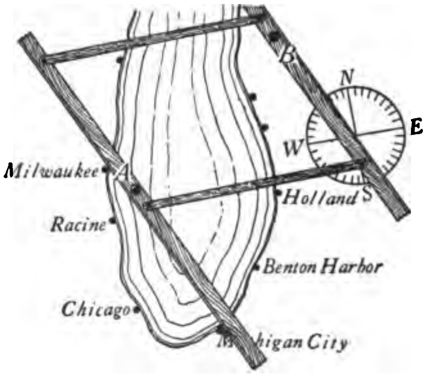


25. The figure shows an instrument called a *parallel ruler*. It is used for drawing parallel lines. It consists of two rulers, AB and DC, which are connected by two cross-pieces, AD and BC. The four parts work on pivots at A, B, C, and D, so that by revolving AD and BC the rulers may be brought close together or placed far apart. $AB = DC$ and $AD = BC$. Prove that for all positions $AB \parallel DC$. Explain how the instrument may be used for drawing parallel lines.



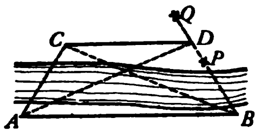
NOTE. — The student would find it easy to make and use a parallel ruler.

26. The parallel ruler is used by sailors in determining the courses of their ships in sailing from one port to another. Thus, to determine the course or direction from Milwaukee to Michigan City on Lake Michigan, the parallel ruler is placed so that ruler A connects these two points on a navigator's map. Explain how the ruler B may then be placed so as to read the required direction on the mariner's compass which is printed on the map, to the east of the lake.

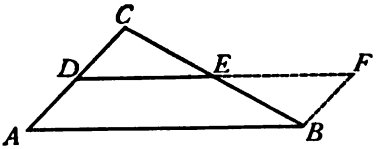


27. Given a line AB on one side of a river and a point C on the other side, surveyors locate a point D such that CD is parallel to AB, as follows: They measure $\angle CAB$ and $\angle CBA$. Then they make $\angle PBA = \angle CAB$, and have an assistant set a stake at P and a second one at Q in line with B and P. A string is stretched from P to Q. Then they make $\angle DAB = \angle CBA$, and set a stake D in line AD, under the string.

Prove that $CD \parallel AB$.



97. Theorem. — *The line-segment which joins the middle points of two sides of a triangle is parallel to the third side and equal to one half of it.*



Hypothesis. In $\triangle ABC$, D and E are the middle points of AC and BC, respectively.

Conclusion. $DE \parallel AB$ and $DE = \frac{1}{2} AB$.

Proof. 1. D and E are the middle points of AC and BC, respectively, *i.e.* $AD = DC$ and $BE = EC$. Hyp.

2. Produce DE through E, and draw $BF \parallel AC$, meeting DE produced at F.

3. $\angle DEC = \angle BEF$. Vert. ∠s

4. $\angle DCE = \angle EBF$. § 29

5. $\therefore \triangle DEC \cong \triangle BEF$. § 65

6. $\therefore BF = DC$ and $EF = DE$. Def. congruence

7. $\therefore BF = AD$. Ax. I

8. $\therefore ABFD$ is a parallelogram. § 90

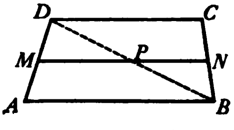
9. $\therefore DE \parallel AB$. Def. ▱

10. Also $DF = AB$. § 82

11. $\therefore DE = \frac{1}{2} AB$. Ax. V

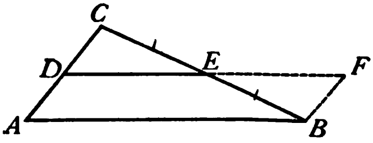
98. Corollary. — *The line-segment connecting the middle points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one half of the sum of the bases.*

Suggestions. Draw DB. Let P be the middle point of DB. Join M, the middle point of AD, to P; and join N, the middle point of BC, to P. Then what relations exist between MP and AB? What relation exists between PN and DC? Then what relation exists between PN and AB? Now show by § 46 that MPN is a straight line, etc.



Write the proof in full.

99. Theorem. — *If a straight line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.*



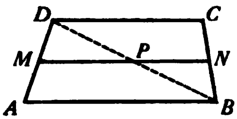
Hypothesis. In $\triangle ABC$, DE bisects AC at D and $DE \parallel AB$.

Conclusion. DE bisects BC.

Suggestions. Produce DE through E, and draw $BF \parallel AC$, meeting DE produced at F. It may be shown that $BE = EC$, if what is proved first? It may be shown that $\triangle BEF \cong \triangle DEC$, if it is first proved that what parts of the triangles are equal? How may these parts be proved equal? Hence begin by proving that $ABFD$ is a parallelogram.

Write out the complete proof.

100. Corollary. — *If a line-segment is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, then it bisects the other also and is equal to one half of the sum of the bases.*

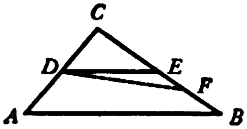


Suggestions. Draw DB, meeting MN at P. Show that MP bisects DB and that $MP = \frac{1}{2} AB$. Then show that PN bisects BC and that $PN = \frac{1}{2} DC$.

Write the proof in full.

EXERCISES

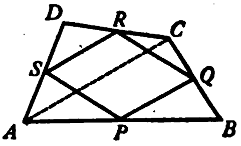
1. Prove the theorem in § 97 by producing DE through E to F, making $EF = DE$, and joining B to F.



2. Prove the theorem in § 99 by drawing a line-segment from E parallel to AC, and meeting AB at F.

3. Prove the theorem in § 99 by the indirect method.

Suggestions. — Suppose that DE does not bisect BC, and that DF does. Then what is true of DF? See § 46.



4. The line-segments connecting the middle points of the adjacent sides of any quadrilateral form a parallelogram.

Suggestion. — In the figure draw diagonal AC. Use § 97.

5. The line-segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.

Suggestion. — Use Ex. 4.

6. The line-segments connecting the middle points of the adjacent sides of a rhombus form a rectangle.

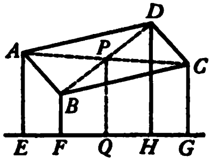
7. The line-segments connecting the middle points of the adjacent sides of a rectangle that is not a square form a rhombus.

8. The line-segments connecting the middle points of the adjacent sides of a square form a square.

9. The quadrilateral which is formed by joining the middle points of the segments into which the diagonals of a given rectangle are divided by their point of intersection is also a rectangle.

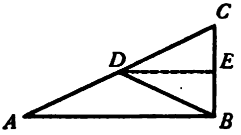
10. Prove by use of § 99 and § 100 that if two lines are both drawn parallel to one side of a triangle, and they divide a second side of the triangle into three equal parts, then they also divide the third side into three equal parts.

11. If perpendiculars AE, BF, CG, DH are drawn to the line EG from the four vertices of ▱ $ABCD$, the sum of the perpendiculars AE and CG equals the sum of the perpendiculars BF and DH.



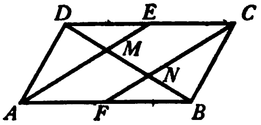
SUGGESTION. — What is the relation of $AE + CG$ to PQ? What is the relation of $BF + DH$ to PQ?

12. In any right triangle, the middle point of the hypotenuse is equidistant from the three vertices.



SUGGESTIONS. — If D is the middle point of the hypotenuse AC of $\triangle ABC$, draw $DE \parallel AB$, meeting BC at E. It may be proved that $DB = DC = DA$ by proving what first? It may be proved that $\triangle DEC \cong \triangle DEB$ by use of what theorem, if it is first proved that BE and EC are equal and that $\angle CED$ and $\angle DEB$ are right angles?

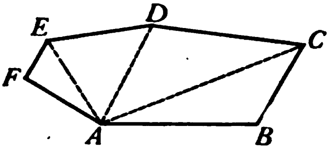
13. $ABCD$ is any parallelogram, and E and F are the middle points of DC and AB, respectively. Prove that AE and CF trisect (cut into three equal parts) DB.



SUGGESTIONS. — It may be shown that $DM = MN$ by § 99 if it is first proved that $ME \parallel NC$. It may be proved that $ME \parallel NC$ if it is first proved that $AFCE$ is a parallelogram. Hence begin by proving that $AFCE$ is a parallelogram.

14. Explain the steps in the construction to trisect a given line-segment by use of the principle in Ex. 13.

101. Theorem. — *The sum of the interior angles of a convex polygon of n sides is (n – 2) straight angles.*



Hypothesis. *ABCD* ··· is a convex polygon of n sides.

Conclusion. The sum of the interior angles of *ABCD* ··· is $(n - 2)$ straight angles.

Proof. 1. Draw all diagonals possible from vertex A, forming $\triangle ABC$, $\triangle ACD$, etc.

2. Since *ABCD* ··· has n sides by hypothesis, there are $(n - 2)$ triangles formed.

(For there is one △ for each side except the two sides adjacent to A.)

3. The sum of the angles of each $\triangle = 1$ st. ∠. § 48

4. ∴ the sum of the angles of all $\triangle$s $= (n - 2)$ st. ∠s. Ax. IV

5. But the sum of the angles of *ABCD* ··· equals the sum of the angles of the triangles. Ax. X

6. ∴ the sum of the angles of *ABCD* ··· $= (n - 2)$ st. ∠s. Ax. I

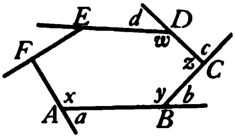
EXERCISES

1. Prove the theorem in § 101 by drawing lines from any point O within the polygon to all of the vertices.

SUGGESTION. — What angles must be subtracted from the sum of the angles of the n triangles to obtain the sum of the angles of the polygon?

2. Find the sum of the angles of a quadrilateral; of a pentagon; of a hexagon; of an octagon; of a decagon; of a polygon of 12 sides; of a polygon of 20 sides.

102. Theorem. — *The sum of the exterior angles of any convex polygon, made by producing each of the sides in succession through a vertex, is 2 straight angles.*



Hypothesis. Polygon $ABCD \cdots$ has each of its n sides produced in succession through a vertex, forming exterior angles a, b, c, d, etc.

Conclusion. $\angle a + \angle b + \angle c + \angle d + \cdots = 2$ st. $\angle$s.

Suggestions. What is the sum of $\angle a$ and $\angle x$? Of $\angle b$ and $\angle y$, etc.? Then what is the sum of all of the exterior and interior angles of the polygon? Since the sum of the interior angles alone is $(n-2)$ st. $\angle$s, find by subtraction the sum of the exterior angles.

Write out the complete proof.

EXERCISES

1. How many degrees has each angle of a regular pentagon? Of a regular hexagon? Of a regular octagon? Of a regular decagon?

2. Show that each angle of a regular polygon of n sides equals $\frac{n-2}{n}$ st. $\angle$s.

3. How many sides has a polygon the sum of whose angles is 14 rt. $\angle$s?

4. How many sides has a regular polygon each of whose angles is 150°? (Form an equation.)

5. How many sides has a regular polygon if each angle is 108°?

6. How many sides has a regular polygon each of whose angles is $1\frac{3}{7}$ rt. $\angle$s?

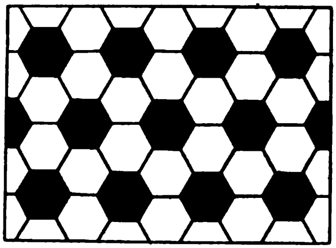
7. The angles of a hexagon are x, x, $\frac{3}{2}x$, $\frac{3}{2}x$, $2x$, and $2x$ degrees. How many degrees are there in each?

8. If two angles of a quadrilateral are supplementary, show that the other two angles must be supplementary also.

9. How many degrees in each exterior angle of a regular hexagon? Of a regular octagon? Of a regular decagon?

10. Show that a floor can be laid by use of tiles in the form of regular hexagons alone. How many such tiles may be placed so as to completely cover the angle about a point, or perigon?

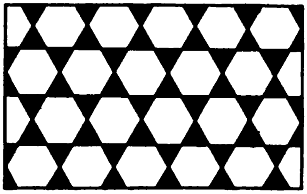
11. Tiles of the forms of what other regular polygons may be used alone for laying floors? Explain why. Why cannot a floor be laid entirely of tiles that are regular pentagons? Regular octagons?



TILE FLOORING

12. What advantage has the bee in always building the cells of its comb in a regular hexagonal form?

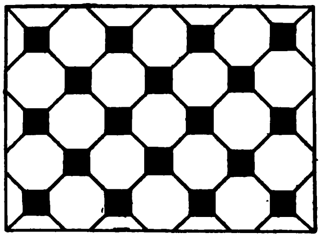
13. In the adjoining figure the tiles of the floor are all regular hexagons and equilateral triangles. Will two such hexagons and two such triangles completely cover the angle about a point, or perigon? Explain.



TILE FLOORING

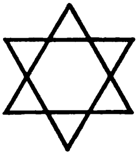
14. In this figure of a linoleum pattern, only regular octagons and squares are used. Prove that two such octagons and a square completely cover the angle about a point.

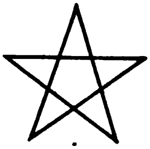
15. Make drawings of other tile floors and linoleum patterns, of parquet flooring, etc., which you have seen, and show how regular polygons are employed in designing them.



LINOLEUM PATTERN

16. How may a six-point star be constructed from a regular hexagon? How many degrees in each point of the star?





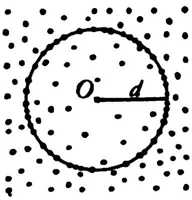
17. How may a five-point star be constructed from a regular pentagon? How many degrees in each point of the star?

Note. — Star polygons, such as those in Ex. 16 and Ex. 17, are used extensively in modern ornament. Almost every piece of cut glass contains one or more of them. The five-point star in Ex. 17 is the one used in the American flag. The use of star polygons dates back to ancient times. The five-point star was used as a symbol of recognition by the members of the Greek school or secret society founded by Pythagoras.

CHAPTER V

LOCI: CONCURRENT LINES

103. A locus of points. — Of all points of a plane, those which are at the same distance *d* from a given point *O* lie on a circle with center *O* and radius *d*. For, by Axiom X, any point not on this circle is at a distance from *O* either greater or less than *d*. Conversely, all points which are on this circle are at the distance *d* from *O*.



This illustrates the general truth that those points of a plane *which possess the same property* are not scattered at random over the plane but are *all located in one or more definite lines*, and also that all points in the line or lines *possess this common property.*

The **locus** (a Latin word meaning *place*) of all points which possess a common property is the line or lines containing those points and no other points. (The plural of *locus* is *loci*, pronounced lō-sī.)

It follows that *the locus of points at a given distance from a given point is a circle with the given point as center and the given distance as radius.*

For discovering what the locus of all points which possess a certain property is, locate, either free-hand or by more careful construction, a number of points which possess the property. These points should suggest the locus, which may be drawn by joining them. This method should be used in the following exercises.

EXERCISES

1. Locate twenty points each 2 in. from the same given point. On what line are they located? That is, what is the locus of such points? Draw it.

2. Locate twenty points each 3 in. from the same given point. What is the locus of such points? Draw it.

3. Locate twenty points each equidistant from the ends of a given line-segment, part on one side and part on the other. What is the locus of such points? Draw it.

4. Within a given angle locate ten points each equidistant from the sides of the angle. How is the distance from a point to a side of the angle represented? What is the locus of such points?

5. Locate ten points each equidistant from two given parallel lines. What is the locus of such points?

6. Locate a number of points each one inch from a given straight line. What is the locus of such points?

7. A point 2 in. from a given point and equidistant from two other given points is at the intersection of what two loci?

104. Proving a locus. — It follows from the definition of a locus of points that to prove that the locus of all points having a given property is a certain line or lines it is necessary *to prove two things*, namely:

(1) *All points on the given line or lines possess the given property.*

(2) *All points which possess the given property are on the given line or lines.*

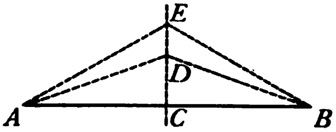
For example, for proving that the locus of points equidistant from the ends of a line-segment is the perpendicular bisector of it, it is necessary to prove that:

(1) All points on the perpendicular bisector are equidistant from the ends of the line-segment;

(2) All points which are equidistant from the ends of the line-segment are on the perpendicular bisector of it.

This is illustrated in the proofs of the following theorems.

105. Theorem. — *The locus of points equidistant from the ends of a line-segment is the perpendicular bisector of it.*



Hypothesis. AB is a given line-segment.

Conclusion. The locus of points equidistant from A and B is the perpendicular bisector of AB.

Proof. 1. Let DC be the perpendicular bisector of AB and let D be any point on DC. Draw AD and BD.

2. $\triangle ACD$ and $\triangle BCD$ are right triangles. Def. $\perp$

3. DC is a common side.

4. $AC = BC$. Def. bisector

5. $\therefore \triangle ACD \cong \triangle BCD$. § 64

6. $\therefore AD = BD$. Def. congruence

7. Hence all points on the perpendicular bisector of AB are equidistant from A and B.

8. Again, let E be any point equidistant from A and B. Join E to C, the middle point of AB, and draw AE and BE.

9. In $\triangle ACE$ and $\triangle BCE$, $AE = BE$ and $AC = BC$. Supposition

10. EC is a common side.

11. $\therefore \triangle ACE \cong \triangle BCE$. § 76

12. $\therefore \angle ACE = \angle BCE$. Def. congruence

13. $\therefore$ But $\angle ACE + \angle BCE =$ st. $\angle$. Def. st. $\angle$

14. $\therefore \angle ACE$ and $\angle BCE$ are right angles. Def. rt. $\angle$

15. $\therefore EC$ is the perpendicular bisector of AB. Def. $\perp$

16. Hence all points equidistant from A and B are on the perpendicular bisector of AB.

17. $\therefore$ the locus of points equidistant from A and B is the perpendicular bisector of AB.

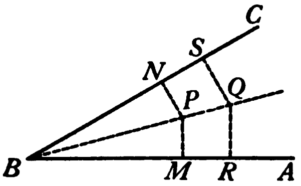
106. Corollary 1. — *A point not on the perpendicular bisector of a line-segment is not equidistant from the ends of it.*

The proof is left to the student.

107. Corollary 2. — *Two points each equidistant from the ends of a line-segment determine the perpendicular bisector of it.*

For both points are on the perpendicular bisector by § 105, and two points determine a straight line.

108. Theorem. — *The locus of points within an angle and equidistant from the sides is the bisector of the angle.*



Hypothesis. $\angle ABC$ is a given angle.

Conclusion. The locus of points within $\angle ABC$ and equidistant from AB and CB is the bisector of $\angle ABC$.

Suggestions. Let PB be the bisector of $\angle ABC$ and let P be any point on PB. Draw $PM \perp AB$ and $PN \perp CB$. Prove $PM = PN$.

Again, let Q be any point within $\angle ABC$ and equidistant from AB and CB. Draw $QR \perp AB$, $QS \perp CB$, and draw QB. Prove that QB bisects $\angle ABC$.

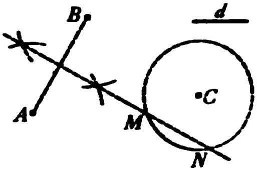
Write the proof in full.

109. Corollary. — *A point within an angle and not on the bisector is not equidistant from the sides of the angle.*

The proof is left to the student.

EXERCISES

1. Locate a point which shall be equidistant from two given points and at a given distance from a third given point.



GIVEN three points, A, B, and C, and a line-segment d.

REQUIRED to locate a point equidistant from A and B and at the distance d from C.

SUGGESTIONS. — On what locus must the required point lie in order to be equidistant from A and B? On what line must it lie in order to be at the distance d from C? Show that either M or N meets the requirements of the point.

Make an accurate construction of your own, and write the construction and proof in full.

Under what condition is the construction impossible?

2. Locate a point which shall be equidistant from the sides of a given angle and at a given distance from a given point.

SUGGESTION. — On what two lines must it lie? It is their intersection. Make the construction and write the discussion in full.

Under what condition is the construction impossible?

3. Locate a point which shall be equidistant from the sides of a given angle and equidistant from two given points.

SUGGESTION. — It is the intersection of what two loci?

Under what condition is the construction impossible?

4. Find a point equidistant from three points not in a straight line.

SUGGESTION. — On what *two* loci does it lie?

5. Find a point within a triangle, equidistant from the three sides.

6. Prove that the locus of points equidistant from two intersecting lines is a pair of lines bisecting the angles formed by these lines.

7. Prove that the locus of points equidistant from two given parallel lines is a line parallel to each of the lines and passing through a point equidistant from them.

8. Prove that the locus of points at a given distance from a given straight line consists of two straight lines parallel to the given line, on opposite sides of it, and at the given distance from it.

9. A point is to be equidistant from two given parallel lines and also equidistant from the sides of a given angle. Locate it.

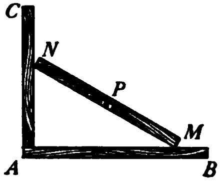
Under what condition is the construction impossible?

10. A point is to be equidistant from two given points and also equidistant from two given parallel lines. Locate it.

Under what condition is the construction impossible?

11. A robber hid some treasure by a roadside and equidistant from two trees. Explain how to find it.

12. Two straightedges, AB and AC, are placed perpendicular to each other. A third straightedge moves with its ends M and N against AB and AC, respectively. Find the locus of all positions taken by the middle point P of MN.

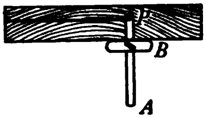


SUGGESTION. — See Ex. 12, page 79.

13. What is the locus of the vertices of all isosceles triangles having the same given base? Prove it.

14. What is the locus of the vertices of all triangles having the same given base and equal medians to this base? Prove it.

15. Carpenters use a tool called a *gauge* for marking a line parallel to the edge of a board. The part A carries a marking point P. The part B may be adjusted on the part A at any required distance from point P by means of a thumb screw. By placing the tool as shown in the figure, with the part B against the edge of a board, and moving the gauge, the point P marks a line on the board parallel to the edge. Why?



110. Concurrent lines. — If three or more lines intersect at one point, they are called **concurrent lines.**

EXERCISES

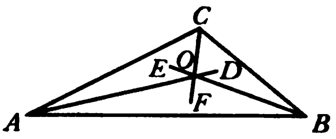
1. Draw any large triangle, and construct accurately with straightedge and compasses the bisectors of the three angles. What is observed about these three bisectors?

2. Similarly, draw a triangle, and construct the perpendicular bisectors of the three sides. What is observed about the three lines?

3. Draw the three altitudes of a triangle. Are they concurrent?

4. Draw the medians of a triangle, and see if they are concurrent.

111. Theorem. — *The bisectors of the three angles of any triangle are concurrent at a point equidistant from the three sides.*



Hypothesis. AD, BE, and CF are the bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively, of $\triangle ABC$.

Conclusion. AD, BE, and CF are concurrent at a point equidistant from AB, AC, and BC.

Proof. 1. AD, BE, and CF bisect $\angle A$, $\angle B$, and $\angle C$, respectively, of $\triangle ABC$. Hyp.

2. $\therefore$ AD and BE intersect at some point O. § 56

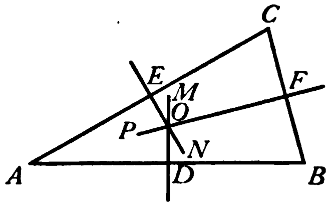
3. Since O is on AD, it is equidistant from AB and AC; since O is on BE, it is equidistant from AB and BC. § 108

4. $\therefore$ O is equidistant from AC and BC. Ax. I

5. $\therefore$ O is on CF; that is, AD, BE, and CF are concurrent at O, which is equidistant from AB, AC, and BC. § 108

NOTE. — The point of intersection of the bisectors of the angles of a triangle is called the incenter of the triangle.

112. Theorem. — *The perpendicular bisectors of the three sides of any triangle are concurrent at a point equidistant from the three vertices.*



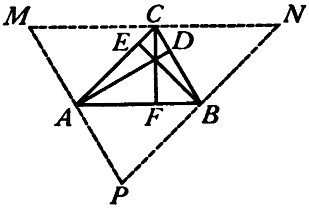
Hypothesis. DM, EN, and FP are the perpendicular bisectors of the sides *AB*, *AC*, and *BC*, respectively, of △ *ABC*.

Conclusion. *DM*, *EN*, and *FP* are concurrent at a point equidistant from *A*, *B*, and *C*.

Suggestions. First prove that *DM* and *EN* intersect at a point *O* by § 55. Then show that *O* is equidistant from *A* and *B* and from *A* and *C*, and hence from *B* and *C*. Why does it follow that *O* is on *FP*?

Note. — The point of intersection of the perpendicular bisectors of the sides of a triangle is called the circumcenter of the triangle.

113. Theorem. — *The three altitudes of any triangle are concurrent.*



Hypothesis. *AD*, *BE*, and *CF* are the altitudes of △ *ABC*.

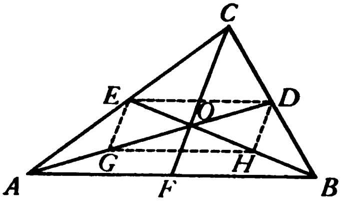
Conclusion. *AD*, *BE*, and *CF* are concurrent.

Suggestions. Draw *MN* ∥ *AB* through *C*, *MP* ∥ *BC* through *A*, and *PN* ∥ *AC* through *B*, forming △ *MNP*. Prove that *AD*, *BE*, and *CF* are perpendicular bisectors of the sides of △ *MNP*. To prove that *C* is the middle point of *MN*, show that $MC = AB = CN$. How may it be proved that *A* is the middle point of *MP* and *B* the middle point of *PN*? Why are the altitudes of △ *ABC* perpendicular to the sides of △ *MNP*? Why then are the altitudes of △ *ABC* concurrent?

Write the proof in full.

Note. — The point of intersection of the altitudes of a triangle is called the orthocenter of the triangle.

114. Theorem. — *The medians of any triangle are concurrent at a point of trisection of each.*



Hypothesis. AD, BE, and CF are the medians of $\triangle ABC$.

Conclusion. AD, BE, and CF are concurrent at a point of trisection of each.

Suggestions. Show by § 56 that AD and BE meet at a point O. Let G and H be the middle points of AO and BO, respectively. Draw GH, HD, DE, and EG. If $GHDE$ is proved a parallelogram, $EO = OH = HB$ and $AG = GO = OD$. Why? Hence, AD and BE meet at a point of trisection of each. Similarly, it may be proved that CF and BE meet at a point of trisection of each, and hence CF passes through O.

Write the proof in full.

Note. — The point of intersection of the medians of a triangle is called the centroid of the triangle.

EXERCISES

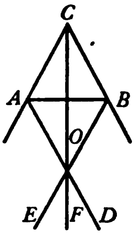
1. Prove that the perpendicular bisectors of the legs of a right triangle intersect on the hypotenuse.

2. Where do the three perpendicular bisectors of the sides of an obtuse triangle intersect? Draw a figure to illustrate.

3. Prove that the altitudes of a right triangle intersect at the vertex of the right angle.

4. Where do the altitudes of an obtuse triangle intersect? Draw a figure to illustrate.

5. Prove that the perpendicular bisectors of the four sides of a rectangle are concurrent.



6. Prove that the perpendicular bisectors of the four sides of an isosceles trapezoid are concurrent.

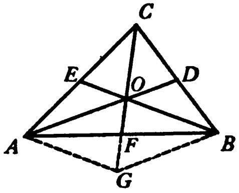
7. Construct a circle which shall pass through the vertices of a given triangle.

8. Prove that the bisectors of the exterior angles at two vertices and the bisector of the interior angle at the third vertex of any triangle are concurrent.

Suggestion. — The proof is similar to that in § 111.

Note. — The point O of intersection is called an *excenter* of the triangle. Every triangle has three excenters.

9. Prove the theorem in § 114 by the following construction: Show that medians AD and BE intersect at a point O. Then draw CF through O, meeting AB at F, and produce it to G, so that $OG = CO$. Draw AG and BG. Show that $AGBO$ is a parallelogram, and hence that $AF = FB$.

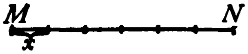


10. If D is the orthocenter of $\triangle ABC$, prove that A is the orthocenter of $\triangle BCD$, B is the orthocenter of $\triangle ACD$, and C is the orthocenter of $\triangle ABD$.

CHAPTER VI

PROPORTION: SIMILAR POLYGONS

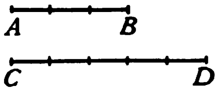
115. Numerical measures of geometric figures. — The **numerical measure** of a geometric figure is the *number of times* that it contains a given unit of measure.



Thus, if the line-segment MN contains the unit of measure x six times, its numerical measure is 6.

Similarly, if $\angle ABC$ contains exactly 30°, its numerical measure, when the unit of measure is a degree, is 30.

116. Ratio of geometric figures. — The **ratio** of two geometric figures of the same kind is the *quotient of their numerical measures*, when the same unit of measure is applied to each.



Thus, if the line-segment AB contains a unit of measure 3 times, and the line-segment CD contains it 5 times, the ratio of AB to CD is 3 divided by 5, or $\frac{3}{5}$.

If one of two angles contains 40° and the other one 60°, the ratio of the first to the second is $\frac{40}{60}$.

In any ratio the dividend is called the **antecedent** and the divisor is called the **consequent**.

117. Proportion. — An expression of equality between two ratios is called a **proportion.**

The proportion $\frac{a}{b} = \frac{c}{d}$, being an equation, is read either as an equation or "a is to b as c is to d."

In any proportion $\frac{a}{b} = \frac{c}{d}$, the numbers a, b, c, and d are called the first, second, third, and fourth **terms**, respectively. The first and fourth terms, a and d, are called the **extremes**. The second and third terms, b and c, are called the **means**.

The four terms of a proportion are said to be "proportional," or "in proportion."

The fourth term of a proportion is called the **fourth proportional** to the other three taken in order. Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$, d is the *fourth proportional* to a, b, and c.

A proportion in which the means are equal is called a **mean proportion**, and either mean term is called the **mean proportional** between the extremes. Thus, in the proportion $\frac{m}{x} = \frac{x}{n}$, x is a *mean proportional* between m and n.

It must be understood from the definition of a proportion that the terms are *abstract numbers*. Hence, since a proportion is an equation, all processes which may be performed upon an equation may be performed upon a proportion.

EXERCISES

Find the value of the unknown term:

1. $\frac{x}{4} = \frac{7}{8}$. **2.** $\frac{6}{v} = \frac{5}{9}$. **3.** $\frac{12}{20} = \frac{n}{5}$. **4.** $\frac{3}{8} = \frac{10}{t}$.

Find the value of n in each proportion:

5. $\frac{a}{b} = \frac{c}{n}$. **6.** $\frac{1}{v} = \frac{v}{n}$. **7.** $\frac{x - n}{x} = \frac{y + n}{y}$. **8.** $\frac{2\pi r}{n} = \frac{2\pi R}{n + r}$.

Find the fourth proportional to:

9. 2, 5, and 6.
10. 3, 8, and 9.
11. 8, 5, and 7.
12. $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{8}$.
13. a, b, and c.
14. 1, $a + b$, and $a - b$.

Find the mean proportional between:

15. 4 and 25.
16. 1 and 36.
17. 12 and 27.
18. 5 and 8.
19. a^2 and b^2.
20. $12x^2$ and $3y^2$.
21. πR^2 and $4\pi R^2$.
22. 4 and $(a - b)^2$.
23. x^2 and $(x + 1)^2$.

118. Fundamental theorems of proportion. — The following fundamental theorems of proportion may be proved by use of axioms. Let the student write the proofs in full.

(1) *If four numbers are in proportion, the product of the extremes equals the product of the means. That is,*

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

Suggestion. Multiply both members by bd.

(2) *If the product of two numbers equals the product of two other numbers, the four numbers are in proportion, one pair of factors being extremes and the other pair means. That is,*

$$\text{if } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

Suggestion. Divide each member by bd.

(3) *If four numbers are in proportion, they are in proportion by inversion. That is,*

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{b}{a} = \frac{d}{c}.$$

Suggestion. Divide 1 by each member.

(4) *In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion. That is,*

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{c} = \frac{b}{d} \text{ and } \frac{d}{b} = \frac{c}{a}.$$

Suggestion. Multiply each member by $\frac{b}{c}$ to prove the first and by $\frac{d}{a}$ to prove the second.

(5) *The terms of any proportion are in proportion by addition. That is,*

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{b} = \frac{c+d}{d}.$$

Suggestion. Add 1 to each member.

(6) *The terms of any proportion are in proportion by subtraction. That is,*

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a-b}{b}=\frac{c-d}{d}.$$

Suggestion. Subtract 1 from each member.

(7) *Like powers or like roots of the terms of a proportion are in proportion. That is,*

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a^n}{b^n}=\frac{c^n}{d^n} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}.$$

Suggestion. Raise each member to the nth power to prove the first, and take the nth root of each member to prove the second.

(8) *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent. That is,*

$$\text{if } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\text{etc., then } \frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=\frac{a}{b}=\text{etc.}$$

For, since the ratios are equal, let each ratio equal r, that is,

$$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r, \text{ etc.}$$

Clearing fractions, $a=rb,\ c=rd,\ e=rf$, etc.

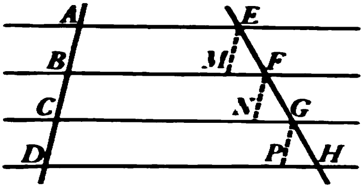
Adding, $a+c+e+\text{etc.}=rb+rd+rf+\text{etc.}$

Factoring, $a+c+e+\text{etc.}=r(b+d+f+\text{etc.})$

Dividing, $\dfrac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=r.$

Hence, by Ax. I, $\dfrac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=\dfrac{a}{b}=\text{etc.}$

119. Theorem. — *If three or more parallel lines cut off equal segments of one transversal, they cut off equal segments of any other transversal.*



Hypothesis. AE, BF, CG, DH, etc., are parallel. They are cut by transversals AD and EH at A, B, C, D, etc., and E, F, G, H, etc., respectively. $AB = BC = CD =$ etc.

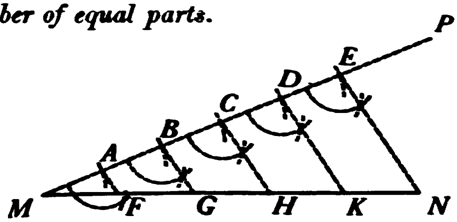
Conclusion. $EF = FG = GH =$ etc.

Suggestions. If EM, FN, GP, etc., are drawn parallel to AD, forming $\triangle MFE$, $\triangle NGF$, $\triangle PHG$, etc., as in the figure, EF, FG, GH, etc., may be proved equal by proving these triangles congruent. For proving these triangles congruent, first prove EM, FN, GP, etc., equal by proving them equal to AB, BC, CD, etc., respectively. Write the proof in full.

120. Corollary. —*If lines parallel to one side of a triangle divide another side into equal segments, they divide the third side also into equal segments.*

The proof is left to the student.

121. Construction. — *Divide a given line-segment into any required number of equal parts.*



Given line-segment MN.

Required to divide MN into any number of equal parts, say five.

Construction. 1. Draw MP, making an angle NMP.

2. On MP mark off 5 equal segments, MA, AB, BC, etc.

3. Join E to N, and through A, B, C, and D, draw parallels to EN, meeting MN at F, G, H, and K.

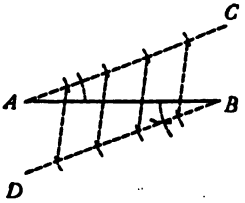
4. Then $MF = FG = GH = HK = KN$.

Proof. The proof is left to the student. Make the construction, and write the discussion in full.

EXERCISES

1. A line-segment AB may be divided into any required number of equal parts, say five, as follows: Draw AC, making a convenient angle BAC, and draw $BD \parallel AC$. Mark off 4 equal distances from A on AC and 4 distances of equal length from B on BD. Connect the points of division as shown in the figure.

Prove that AB is divided into 5 equal parts.

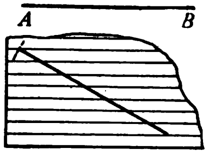


2. Divide a line-segment into 3 equal parts.

3. Divide a line-segment into 7 equal parts.

4. Divide a line-segment into 10 equal parts.

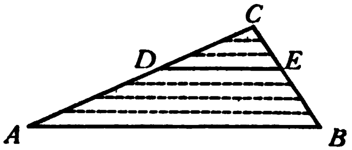
5. Prove that by placing a line-segment AB over a piece of ruled or squared paper, as in the figure, it may be divided into any number of equal parts, say 7.



6. Divide a given line-segment into 3 equal parts, using ruled paper. Into 5 equal parts. Into 8 equal parts.

7. A board is to be ripped into three strips of equal width. A carpenter places his 12-inch ruler as shown in the drawing, and marks points on the board at the 4-inch and 8-inch marks. He then rips the board by sawing lengthwise through these points. Prove that the strips are of equal width.

122. Theorem. — *A line parallel to one side of a triangle divides the other two sides proportionally.*



Hypothesis. In $\triangle ABC$, $DE \parallel AB$ and DE meets AC at D and BC at E.

Conclusion. $\frac{DC}{AD} = \frac{EC}{BE}$.

Proof. 1. Suppose that a common unit of measure may be found that is contained in DC and AD m and n times, respectively.

2. Then lines drawn through the points of division of AC, parallel to AB, would divide BC into $m + n$ equal parts, m parts in EC, and n parts in BE. §120

3. $\therefore \frac{DC}{AD} = \frac{m}{n}$ and $\frac{EC}{BE} = \frac{m}{n}$. Def. ratio

4. $\therefore \frac{DC}{AD} = \frac{EC}{BE}$. Ax. I

NOTE. — The proof above is incomplete theoretically, since it assumes that DC and AD have a common unit of measure, which need not be true. There are line-segments, such as the side and diagonal of a square, which have no common unit of measure. The theorem is equally true, however, when no common unit of measure of AD and DC can be found. The proof for that case is omitted from the elementary course in geometry.

123. Corollary. — *If DE is parallel to side AB of triangle ABC and meets AC at D and BC at E, then*

$$\frac{AC}{AD} = \frac{BC}{BE} \quad \textit{and} \quad \frac{AC}{DC} = \frac{BC}{EC}.$$

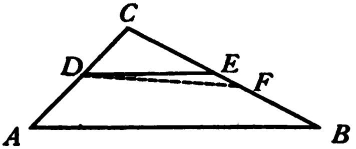
Proof. 1. $\frac{DC}{AD}=\frac{EC}{BE}$. § 122

2. $\therefore$ $\frac{DC+AD}{AD}=\frac{EC+BE}{BE}$, or $\frac{AC}{AD}=\frac{BC}{BE}$. § 118, (5)

3. From step 1, $\frac{AD}{DC}=\frac{BE}{EC}$. § 118, (3)

4. $\therefore$ $\frac{AD+DC}{DC}=\frac{BE+EC}{EC}$, or $\frac{AC}{DC}=\frac{BC}{EC}$. § 118, (5)

124. Theorem. — *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*



Hypothesis. In $\triangle ABC$, DE meets AC at D and BC at E, such that $\frac{DC}{AD}=\frac{EC}{BE}$.

Conclusion. $DE \parallel AB$.

Proof. 1. Suppose DE not parallel to AB, but $DF \parallel AB$.

2. Then $\frac{DC}{AD}=\frac{FC}{BF}$. § 122

3. But $\frac{DC}{AD}=\frac{EC}{BE}$. Hyp.

4. $\therefore$ $\frac{FC}{BF}=\frac{EC}{BE}$. Ax. I

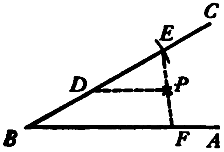
5. $\therefore$ $\frac{FC+BF}{BF}=\frac{EC+BE}{BE}$, or $\frac{BC}{BF}=\frac{BC}{BE}$. § 118, (5)

6. Clearing fractions and dividing by BC, $BF=BE$.

7. But this is impossible. Ax. X

8. $\therefore$ the supposition is false, and hence $DE \parallel AB$.

8. Having given a point P within $\angle ABC$, draw a line through P so that the segment of it lying within the angle shall be bisected at P.



SUGGESTION. — If EP is to equal PF, what relation must exist between BD and DE? Hence begin by drawing $PD \parallel AB$.

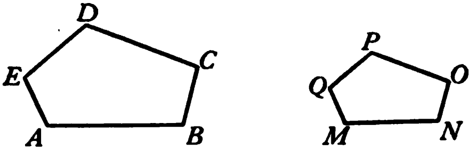
9. Having given a point P within $\angle ABC$, draw a line through P so that the segment of it lying within the angle shall be divided at P in the ratio of 1 to 2.

10. Having given a point P within $\angle ABC$, draw a line through P so that the segment of it lying within the angle shall be divided at P in the ratio of m to n.

127. Similar polygons.—Two polygons which have the angles of one equal respectively to the angles of the other, taken in order, are called **mutually equiangular**.

The pairs of equal angles are called **corresponding angles**. The sides included between corresponding angles are called **corresponding sides**. Corresponding sides are also called **homologous sides**.

Similar polygons are those which (1) are mutually equiangular and (2) have their corresponding sides proportional.



SIMILAR POLYGONS

Thus polygons $ABCDE$ and $MNOPQ$ are similar if

(1) $\angle A = \angle M$, $\angle B = \angle N$, $\angle C = \angle O$, $\angle D = \angle P$, $\angle E = \angle Q$,

(2) $$\frac{AB}{MN} = \frac{BC}{NO} = \frac{CD}{OP} = \frac{DE}{PQ} = \frac{EA}{QM}.$$

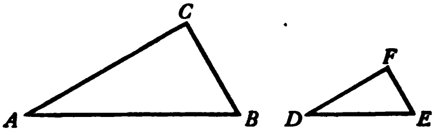
Similarity of polygons is expressed by the symbol $\sim$.

Thus "*ABCDE* is similar to *MNOPQ*" is written "$ABCDE \sim MNOPQ$."

It is apparent that similar polygons are of *like form* or *shape*, which is the fundamental idea of similarity.

EXERCISES

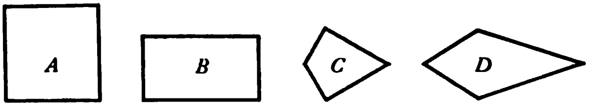
1. Draw any triangle ABC. Construct a second triangle DEF with $DE = \frac{1}{2} AB$, $\angle D = \angle A$, and $\angle E = \angle B$. Why are the triangles mutually equiangular?



Measure AC, BC, DF, and EF. Divide the measures thus obtained so as to get the ratios $\frac{AB}{DE}$, $\frac{BC}{EF}$, and $\frac{AC}{DF}$. What is observed about the ratios $\frac{AB}{DE}$, $\frac{BC}{EF}$, and $\frac{AC}{DF}$?

It thus appears that if any two triangles are mutually equiangular, their corresponding sides are in proportion, and conversely, if their corresponding sides are in proportion, they are mutually equiangular. That is, either condition for similarity implies the other.

2. Polygons A and B are mutually equiangular. Are their corresponding sides proportional?

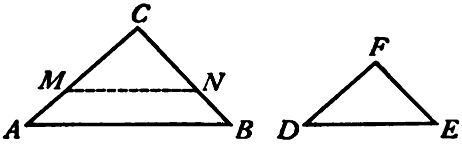


Polygons C and D have their sides proportional. Are they mutually equiangular?

It thus appears that in the case of polygons of more than three sides neither of the conditions for similarity implies the other.

NOTE. — It follows from Ex. 2 that to prove any two polygons similar, both conditions for similarity must be established. In the following theorems it will be proved that in the case of triangles, as suggested in Ex. 1, either condition for similarity is sufficient to make the triangles similar.

128. Theorem.—*If two triangles are mutually equiangular, they are similar.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

Conclusion. $\triangle ABC \sim \triangle DEF$.

Proof. 1. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$. Hyp.

2. $\therefore \triangle DEF$ may be placed so that $\angle F$ coincides with $\angle C$, $\triangle DEF$ taking the position of $\triangle MNC$. § 13

3. Then $MN \parallel AB$. § 36

4. $\therefore \dfrac{AC}{MC} = \dfrac{BC}{NC}$. § 123

5. $\therefore \dfrac{AC}{DF} = \dfrac{BC}{EF}$. Ax. XII

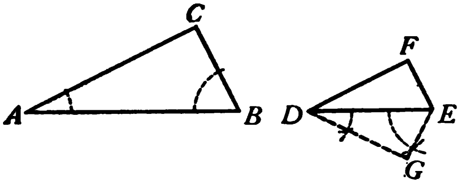
6. Similarly, $\dfrac{AC}{DF} = \dfrac{AB}{DE}$ and $\dfrac{BC}{EF} = \dfrac{AB}{DE}$.

7. $\therefore \triangle ABC \sim \triangle DEF$. Def. sim. poly.

For proving $\dfrac{AC}{DF} = \dfrac{AB}{DE}$, how must $\triangle DEF$ be placed upon $\triangle ABC$?

For proving $\dfrac{BC}{EF} = \dfrac{AB}{DE}$, how must $\triangle DEF$ be placed upon $\triangle ABC$?

129. Theorem.—*If two triangles have their corresponding sides proportional, the triangles are similar.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$.

Conclusion. $\triangle ABC \sim \triangle DEF$.

Proof. 1. Construct $\angle EDG=\angle A$ and $\angle DEG=\angle B$, forming $\triangle DEG$.

2. Then $\angle C=\angle G$. § 49

3. $\therefore \triangle ABC \sim \triangle DEG$. § 128

4. $\therefore \frac{AB}{DE}=\frac{BC}{EG}$. Def. sim. poly.

5. But $\frac{AB}{DE}=\frac{BC}{EF}$. Hyp.

6. $\therefore \frac{BC}{EG}=\frac{BC}{EF}$. Ax. I

7. $\therefore EF=EG$. Ax. IV and Ax. V

8. Similarly, $DF=DG$.

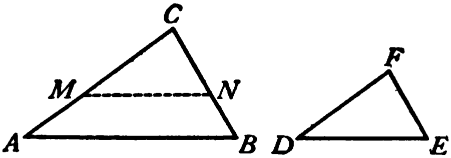
9. $\therefore \triangle DEF \cong \triangle DEG$. § 76

10. $\therefore \angle EDF=\angle EDG$, $\angle DEF=\angle DEG$, $\angle F=\angle G$. Def. congruence

11. $\therefore \angle EDF=\angle A$, $\angle DEF=\angle B$, $\angle F=\angle C$. Ax. I

12. $\therefore \triangle ABC \sim \triangle DEF$. Def. sim. poly.

130. Theorem. — *If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, they are similar.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\angle C = \angle F$ and $\frac{AC}{DF} = \frac{BC}{EF}$.

Conclusion. $\triangle ABC \sim \triangle DEF$.

Proof. 1. $\angle C = \angle F$, and $\frac{AC}{DF} = \frac{BC}{EF}$. Hyp.

2. $\therefore \triangle DEF$ may be placed upon $\triangle ABC$, taking the position of $\triangle MNC$. § 13

3. Also $\frac{AC}{MC} = \frac{BC}{NC}$. Ax. XII

4. $\therefore \frac{AC - MC}{MC} = \frac{BC - NC}{NC}$, or $\frac{AM}{MC} = \frac{BN}{NC}$. § 118 (6)

5. $\therefore \frac{MC}{AM} = \frac{NC}{BN}$. § 118 (3)

6. $\therefore MN \parallel AB$. § 124

7. $\therefore \angle CMN = \angle A$ and $\angle CNM = \angle B$. § 26

8. But $\angle CMN = \angle D$ and $\angle CNM = \angle C$. Construction

9. $\therefore \angle A = \angle D$ and $\angle B = \angle E$. Ax. I

10. $\therefore \triangle ABC \sim \triangle DEF$. § 128

EXERCISES

1. Two triangles are similar if they have two pairs of equal angles.

2. Two right triangles are similar if they have a pair of equal acute angles.

3. Two isosceles triangles are similar if the vertical angles are equal.

4. If a line is drawn parallel to one side of a triangle, intersecting the other two sides, it cuts off a triangle similar to the given triangle.

5. The altitudes to the corresponding sides of similar triangles are proportional to any two corresponding sides.

6. The medians to the corresponding sides of two similar triangles are proportional to any two corresponding sides.

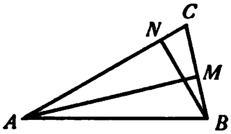
7. The bisectors of two corresponding angles of similar triangles have the same ratio as any two corresponding sides.

8. If two triangles have the sides of one respectively parallel to the sides of the other, they are similar.

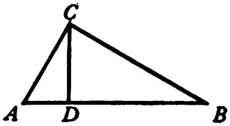
9. If two triangles have the sides of one respectively perpendicular to the sides of the other, they are similar.

10. If each of two triangles is similar to a third triangle, they are similar to each other.

11. If AM and BN are altitudes of $\triangle ABC$, $\triangle AMC \sim \triangle BNC$.



12. In $\triangle ABC$, $\angle C$ is a right angle and CD is perpendicular to AB, the hypotenuse. Prove $\triangle ABC \sim \triangle ADC$, $\triangle ABC \sim \triangle DBC$, and $\triangle ADC \sim \triangle DBC$.



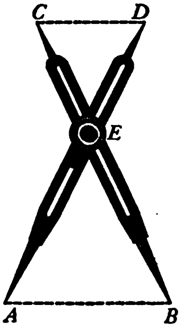
13. The sides of a triangle are 2 in., 3 in., and 4 in. The shortest side of a similar triangle is 5 in. Find the other two sides.

14. The diagonals of a trapezoid divide each other into proportional segments.

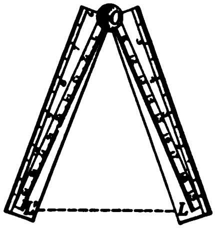
15. In trapezoid $ABCD$, with bases AB and DC, the diagonals AC and BD meet at E. If $AE = 12$, $EC = 8$, and $BD = 24$, find BE and ED.

16. The instrument shown in the drawing is a draftsman's pair of *proportional compasses.* The lengths AE, BE, ED, and EC are adjusted proportionally by means of the screw at E. Prove that $\frac{AB}{CD} = \frac{AE}{ED}$.

If $AE = 6$ in. and $ED = 2$ in., the distance between A and B is how many times the distance between C and D?



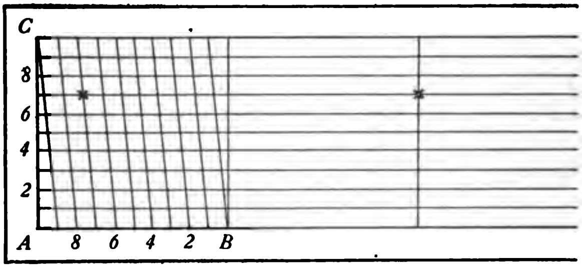
17. The instrument represented in the figure is called a *sector*. By means of it various constructions and measurements can be made. Thus, to bisect a line-segment, open the sector until the transverse distance from 10 to 10 on the scales OL and OL' equals the given segment. Then the distance from 5 to 5 on these scales is equal to one half of the segment. Give the proof.



18. The sector may be used to divide a line-segment into any number of equal parts, say six, as follows: Open the sector until the distance from 6 to 6 on the scales equals the given segment. Then the transverse distance from 1 to 1 will be one sixth of the given segment. Give the proof.

19. Show how by use of the sector to divide a given line-segment into 9 equal parts.

20. The instrument shown in this drawing is a *diagonal scale*, used by draftsmen for measuring very short lengths. $AB = 1$ in. AB and AC are each divided into 10 equal parts, and parallels are drawn through the points of division as shown. By means of this scale distances can be measured to hundredths of an inch.



Find .01 in. on this scale; .02 in.; .03 in.; .04 in.; etc.

Upon what theorem does this depend?

21. Find .12 in. on the diagonal scale; .35 in.; .93 in.; .68 in.

22. What is the distance between the points marked X- - - - - - - - - - -X on the diagonal scale?

23. Find 1.63 in. on the diagonal scale; 1.85 in.; 1.19 in.

24. Draw a triangle and measure the lengths of its sides to hundredths of an inch by use of the diagonal scale. In measuring, apply the compasses to a side of the triangle, then to the diagonal scale.

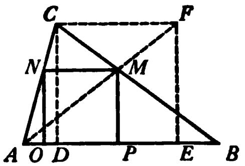
25. Draw a square with a side 1 in. long. Draw a diagonal. By use of compasses and diagonal scale, measure the length of the diagonal to hundredths of an inch. Find the ratio of the diagonal to a side of the square.

26. Draw a line-segment 1.23 in. long. One 1.07 in. long. One 0.89 in. long.

27. Construct an equilateral triangle whose side is 1.37 in.

28. Construct a triangle whose sides are 1.62 in., 1.74 in., and 1.95 in.

29. The square *OPMN* is said to be inscribed in $\triangle ABC$. By studying the figure, explain how to construct the square *OPMN* in the given triangle, and give the proof.



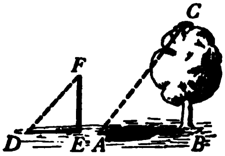
SUGGESTION. — *DEFC* is a square, and *AF* meets *BC* at *M*, determining one vertex of the required square. Prove $\frac{NM}{CF}$ and $\frac{PM}{EF}$ equal to the same ratio, and hence show that $NM = PM$.

131. The practical measurement of distances. — The proportion between the sides of similar triangles affords a useful means of finding distances in practical work where direct measurements are impossible. Some applications are shown in the following problems.

EXERCISES

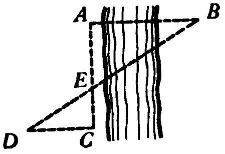
1. The fact that the heights of two objects are to each other as the lengths of their shadows has been used since the time of the ancient Greeks. Thales (about 600 B.C.) is said to have amazed the Egyptians by measuring the heights of the pyramids by the lengths of their shadows.

Give a proof of this principle.



2. A stick EF held vertically casts a shadow DE which is $3\frac{1}{2}$ ft. long. At the same time a tree BC casts a shadow AB which is 50 ft. long. If $EF = 6$ ft., find BC.

3. The distance AB across a stream is obtained as follows: A line AC is run off at right angles to AB, along the shore. From C the line CD is measured off at right angles to AC. The point E of AC which is in line with D and B is then located. AE and EC are measured. AB is then computed by proportion.

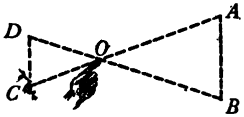


Write the proportion between AB and the measured distances. Prove this proportion.

4. In the figure of Ex. 3, if $AE = 120$ ft., $EC = 30$ ft., and $CD = 40$ ft., find AB.

5. The following method may be used for estimating the distance from the observer to an inaccessible object:

With the left eye closed, the finger is pointed, at arm's length (at O), toward the object A. Then without moving the finger, the right eye is closed and left eye opened, when the object appears to have moved to B. The distance AB through which it appears to have moved, being transverse to the line of sight, is estimated. The distance from the finger O to the object is approximately 10 times the distance AB.

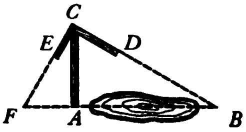


The distance CO from the eye to the outstretched finger of the average person is approximately 10 times the distance CD between the eyes. Prove, then, that $OA = 10\,AB$.

If the object appears to have moved 1100 ft., how far away is it? How many miles?

6. The U. S. S. *Texas* (1915) is 565 ft. long. When observed at sea by the method of Ex. 5, it appeared to move through a distance of four ship lengths. How far away was it? How many miles?

7. A method employed several centuries ago, before modern instruments were invented, for determining the distance from A to an inaccessible point B was as follows: Upon a vertical staff AC was placed an instrument resembling a carpenter's square. The blade CD was pointed toward B, and at the same time the point F on the ground at which the blade CE



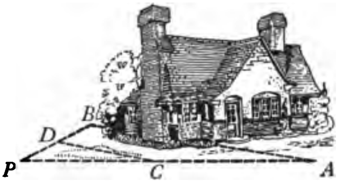
pointed was marked. FA and AC were measured. Then AB was computed by the proportion $\frac{FA}{AC}=\frac{AC}{AB}$.

Prove this proportion.

8. In Ex. 7, if $FA = 2\frac{3}{8}$ in., and $AC = 62$ in., find AB.

Note. — The student would find it interesting and profitable to use the different methods suggested in the preceding exercises in determining the heights and distances of objects or points in the neighborhood of the school.

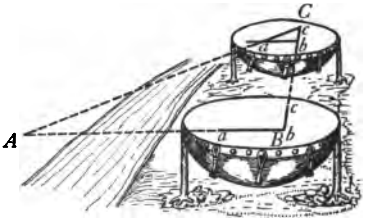
9. The distance between two accessible points A and B which are separated by an obstacle may be measured as follows: From a convenient point P the distances PA and PB are measured. Then in these lines points C and D, respectively, are located such that $\frac{PC}{PA}=\frac{PD}{PB}$. Finally, CD is measured.



Prove the proportion by means of which AB may now be computed.

If $PA = 840$ ft., $PC = 120$ ft., $CD = 400$ ft., find AB.

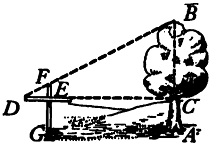
10. Before modern instruments were invented, an inaccessible distance AB was measured by use of *drum heads* as follows: On a drum head placed at B a line ba was drawn toward A and a line bc toward an accessible point C. BC was then measured, and the drum head removed to C and placed with bc in the direction BC, as indicated. Then a third line ca was drawn toward A. Show how it was possible from these measurements to compute AB. If $BC = 200$ yd., $bc = 10$ in., and $ba = 16$ in., compute AB.



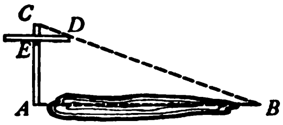
11. The *cross-staff* (See Ex. 6 and Note, page 53) may be used to find the height AB of an object as follows: The horizontal cross-bar DE is raised or lowered on the staff FG until D, F, and B fall in a straight line. Then DE, EF, GE, and GA are measured.

Explain how AB may be computed.

If $DE = 18$ in., $EF = 24$ in., $GE = 36$ in., and $GA = 60$ ft., find AB.

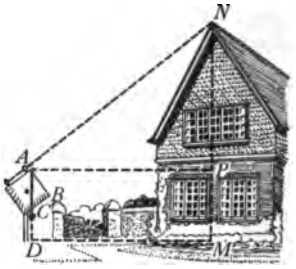


12. By the aid of the diagram explain how the cross-staff may be used to obtain the horizontal distance from a point A to an inaccessible point B.



If $AC = 60$ in., $EC = 1\frac{3}{4}$ in., and $ED = 18$ in., find AB.

13. The *geometric square* is another instrument used in practical measurements before modern engineering instruments were invented. It consists of a square frame, along two adjacent edges of which is marked a scale, and from the opposite corner of which a plumb line is suspended. A pair of sights on another edge aid in pointing the instrument.

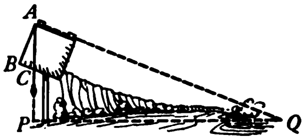


When the height MN of an object is to be found, the square is held in a vertical plane and the edge bearing the sights is pointed toward N. The point C where the plumb line then crosses the scale is noted. The height of the instrument DA and the horizontal distance DM are measured. PN is computed by proportion.

Prove that $\frac{AB}{AP} = \frac{CB}{PN}$.

14. In the figure of Ex. 13, if $AB = 48$, $CB = 20$, $DM = 96$ ft., and $DA = 5$ ft., find MN.

15. The horizontal distance PQ may be measured by means of the geometric square as follows: The square is held directly over P and the edge bearing the sights directed toward Q. The point C where the plumb line then crosses the scale is noted. The height of the instrument AP is measured. Then PQ is computed by proportion.

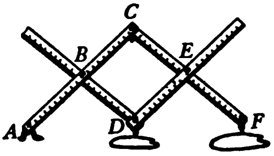


Prove the proportion by means of which PQ may be computed.

If $AP = 4\frac{3}{4}$ ft., $AB = 48$, and $BC = 1\frac{1}{2}$, find PQ.

Note. — It will be found easy and interesting for the student to make a geometric square and use it in such measurements as those described in the exercises above.

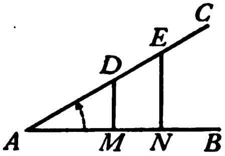
16. A *pantograph* is an instrument for drawing a figure similar to a given figure, and is useful for enlarging or reducing maps and drawings. It consists of four bars, parallel in pairs and jointed at B, C, D, and E. A turns on a fixed pivot, and pencils are carried at D and F. BD and DE are so adjusted as to form a parallelogram $BCED$ and such that any required ratio $\frac{AB}{AC}$ is equal to $\frac{CE}{CF}$.



Show that A, D, and F are always in a straight line.

17. Show that in the pantograph of Ex. 16 the ratio $\frac{AD}{AF}$ remains constant and equal to $\frac{AB}{AC}$, so that if the pencil F traces a given figure, the pencil D will trace a similar figure.

132. Trigonometric ratios. — Let $\angle BAC$ be any acute angle. Mark off any two distances AD and AE on AC. Draw $DM \perp AB$ and $EN \perp AB$.



Then $\triangle AMD \sim \triangle ANE$. Why?

Hence, $\frac{DM}{EN} = \frac{AD}{AE}$. Why?

Therefore, (1) $\frac{DM}{AD} = \frac{EN}{AE}$. Why?

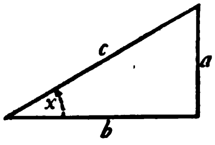
Similarly, (2) $\frac{AM}{AD} = \frac{AN}{AE}$. Explain.

And (3) $\frac{DM}{AM} = \frac{EN}{AN}$. Explain.

That is, the ratios of the (1) *altitude to the hypotenuse*, (2) *base to the hypotenuse*, and (3) *altitude to the base* are the same regardless of the length marked off along the side AC. This is expressed by saying that *for any given* $\angle BAC$, *the three ratios are constant.*

These three constant ratios which are connected with any given acute angle are called **trigonometric ratios.**

133. Naming trigonometric ratios. — Each of the three constant ratios between the sides of a right triangle that is formed with one of its acute angles equal to a given angle is given a special name.



Let a, b, and c, as shown in the figure, be the sides of a right triangle formed with the given acute angle x. Then

$\frac{a}{c}$ is called the **sine** of $\angle x$, written *sin x*,

$\frac{b}{c}$ is called the **cosine** of $\angle x$, written *cos x*,

$\frac{a}{b}$ is called the **tangent** of $\angle x$, written *tan x*.

134. Trigonometric tables. — If a right triangle is accurately constructed, using a protractor, with an acute angle of 40°, and if the sides of this triangle are measured and the ratios of a to c, b to c, and a to b, computed to four decimal places by division, it will be found that

$$\sin 40^\circ = \frac{a}{c} = .6428,$$

$$\cos 40^\circ = \frac{b}{c} = .7660,$$

$$\tan 40^\circ = \frac{a}{b} = .8391.$$

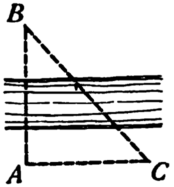
Other methods may be employed for computing the trigonometric ratios of angles. Thus, if the angle is 45°, the triangle is isosceles, and hence $\tan 45^\circ = 1$. If the angle is 30°, $a = \frac{1}{2}c$, and hence $\sin 30^\circ = \frac{1}{2}$ or .5000.

The values of the trigonometric ratios of angles have been accurately computed. The following table gives these values for all integral numbers of degrees from 1° to 89°, correct to four decimal places.

VALUES OF SINES, COSINES, AND TANGENTS

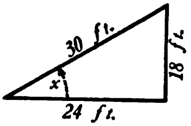
Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.0175	.9998	.0175	45	.7071	.7071	1.0000
2	.0349	.9994	.0349	46	.7193	.6947	1.0355
3	.0523	.9986	.0524	47	.7314	.6820	1.0724
4	.0698	.9976	.0699	48	.7431	.6691	1.1106
5	.0872	.9962	.0875	49	.7547	.6561	1.1504
6	.1045	.9945	.1051	50	.7660	.6428	1.1918
7	.1219	.9925	.1228	51	.7771	.6293	1.2349
8	.1392	.9903	.1405	52	.7880	.6157	1.2799
9	.1564	.9877	.1584	53	.7986	.6018	1.3270
10	.1736	.9848	.1763	54	.8090	.5878	1.3764
11	.1908	.9816	.1944	55	.8192	.5736	1.4281
12	.2079	.9781	.2126	56	.8290	.5592	1.4826
13	.2250	.9744	.2309	57	.8387	.5446	1.5399
14	.2419	.9703	.2493	58	.8480	.5299	1.6003
15	.2588	.9659	.2679	59	.8572	.5150	1.6643
16	.2756	.9613	.2867	60	.8660	.5000	1.7321
17	.2924	.9563	.3057	61	.8746	.4848	1.8040
18	.3090	.9511	.3249	62	.8829	.4695	1.8807
19	.3256	.9455	.3443	63	.8910	.4540	1.9626
20	.3420	.9397	.3640	64	.8988	.4384	2.0503
21	.3584	.9336	.3839	65	.9063	.4226	2.1445
22	.3746	.9272	.4040	66	.9135	.4067	2.2460
23	.3907	.9205	.4245	67	.9205	.3907	2.3559
24	.4067	.9135	.4452	68	.9272	.3746	2.4751
25	.4226	.9063	.4663	69	.9336	.3584	2.6051
26	.4384	.8988	.4877	70	.9397	.3420	2.7475
27	.4540	.8910	.5095	71	.9455	.3256	2.9042
28	.4695	.8829	.5317	72	.9511	.3090	3.0777
29	.4848	.8746	.5543	73	.9563	.2924	3.2709
30	.5000	.8660	.5774	74	.9613	.2756	3.4874
31	.5150	.8572	.6009	75	.9659	.2588	3.7321
32	.5299	.8480	.6249	76	.9703	.2419	4.0108
33	.5446	.8387	.6494	77	.9744	.2250	4.3315
34	.5592	.8290	.6745	78	.9781	.2079	4.7046
35	.5736	.8192	.7002	79	.9816	.1908	5.1446
36	.5878	.8090	.7265	80	.9848	.1736	5.6713
37	.6018	.7986	.7536	81	.9877	.1564	6.3138
38	.6157	.7880	.7813	82	.9903	.1392	7.1154
39	.6293	.7771	.8098	83	.9925	.1219	8.1443
40	.6428	.7660	.8391	84	.9945	.1045	9.5144
41	.6561	.7547	.8693	85	.9962	.0872	11.4301
42	.6691	.7431	.9004	86	.9976	.0698	14.3006
43	.6820	.7314	.9325	87	.9986	.0523	19.0811
44	.6947	.7193	.9657	88	.9994	.0349	28.6363
45	.7071	.7071	1.0000	89	.9998	.0175	57.2900

135. Finding distances and angles by trigonometric ratios. — The trigonometric ratios are used in practical work for the indirect measurement of distances and angles.



Thus, to find the distance from A to the inaccessible point B, an engineer measures a line AC to a convenient point C, making $\angle A$ a right angle. Then he measures $\angle C$. If $AC = 650$ ft. and $\angle C = 48°$, then $\dfrac{AB}{650 \text{ ft.}} = \tan 48° = 1.1106$. Hence $AB = 1.1106 \times 650 \text{ ft.} = 721.89$ ft.

EXERCISES



1. In the figure, find $\sin x$, $\cos x$, $\tan x$.

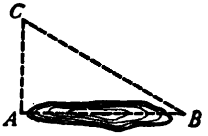
2. Construct an angle whose tangent is $\frac{3}{4}$.

3. Construct an angle whose sine is $\frac{2}{3}$.

4. Construct an angle whose cosine is $\frac{5}{8}$.

5. Obtain from the table sin 75°; cos 16°; tan 55°.

6. Find from the table the angle x if $\sin x = .3256$. If $\cos x = .1219$. If $\tan x = .4663$.



7. In order to find the distance from A to B across a lake, a surveyor measured a line AC at right angles to AB, then measured $\angle BCA$. If $AC = 820$ ft. and $\angle BCA = 56°$, find AB.

8. The distance from the observer to the foot of a monument is 275 ft. The angle of elevation of the monument at the point of observation is 52°. Find the height.

9. A building known to be 136 ft. high forms an angle of elevation of 23° at a point of observation. How far is the observer from the building?

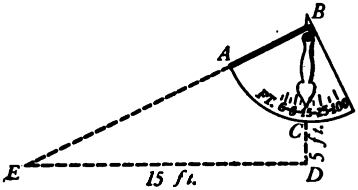
10. A vertical rod 8 ft. high casts a shadow 3 ft. 6 in. long. Find the angle of elevation of the sun.

11. The distance from the base to the top of a hill, up a uniform incline of 40°, is 300 yd. What is the altitude of the top above the base?

12. A kite string 1000 ft. long makes an angle of 60° with horizontal. How high is the kite, not allowing for sag in the string? It is directly over a spot how far from the holder of the string?

13. A house 32 ft. wide has a gable with rafters 22 ft. long, excluding the part which projects below the eaves. Find the *pitch* of the roof, or the angle between a rafter and horizontal.

14. The instrument called a *telemeter*, shown in the drawing, is used by photographers for estimating distances when taking pictures. AB is pointed toward the object E to which the distance is to be found. The plumb line points on the scale to the number of feet that the object E is distant. The instrument is made to be held at a height BD of 5 ft. above the ground.

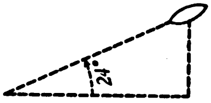


In making the scale of feet on the telemeter, the 15-ft. mark of the scale must be placed at a point C so that $\angle ABC$ is how many degrees?

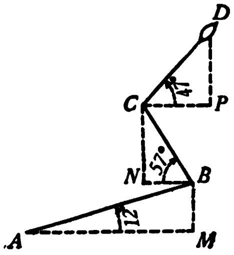
Compute in the same way the number of degrees in $\angle ABC$ when C is at the 6-ft. mark. When C is at the 25-ft. mark.

15. A mariner finds that the angle of elevation of a light from a lighthouse is 7°. The lighthouse is known to stand 60 ft. above the level of the ship. How far is his ship from the lighthouse?

16. A ship is sailing from New York in a direction 24° north of east. When the log shows that it has gone 450 mi., how far is the ship east of New York? How far north?

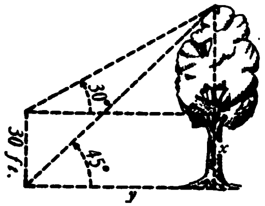


17. A sailing ship, tacking against the wind, sails from a port A to B along a course 12° N. of E., from B to C along a course 57° N. of W., then from C to D along a course 47° N. of E. The log shows that the ship has gone the following distances: $AB = 72$ mi., $BC = 38$ mi., $CD = 40$ mi. The mariner wishes to know, when at D, how far he is east and how far he is north of A. Compute these distances.



SUGGESTION. — Compute AM, BN, and CP. Then add AM and CP and subtract BN from the result to find the distance that D is east of A. The distance that D is north of A may be found similarly by computing BM, CN, and DP.

18. The angle of elevation of a tree is 45° from a point on the ground, and from an upper window, 30 ft. from the ground, the angle of elevation is only 30°. How high is the tree?

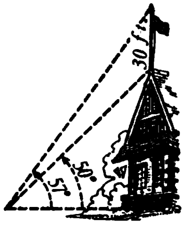


SUGGESTION. — Let x be the required height. The figure gives the equations:

$$x - 30 = y \tan 30^\circ,$$
$$x = y \tan 45^\circ.$$

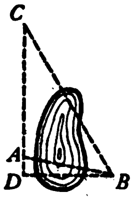
Solve for x.

19. A 30-ft. flagpole is mounted on the roof of a building. The angles of elevation above horizontal of the top and bottom of the pole are 57° and 50°, respectively. Find the height of the building.



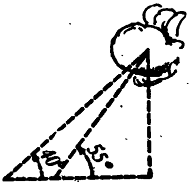
SUGGESTION. — Form two equations in two unknown numbers. An equation may be obtained from each right triangle.

20. To find the distance from A to B, across a lake, being unable to measure perpendicularly to AB, I measured 120 yd. to C, making $\angle CAB = 100^\circ$, and observed that $\angle ACB = 30^\circ$. Find the distance from A to B.



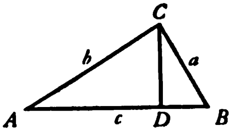
SUGGESTION. — First find AD or DB. Then from this find AB. For finding AD or DB, form two equations in two unknown numbers. From what two triangles may they be obtained?

21. How high is a cloud when two observers, so placed as to be in a vertical plane with the cloud, and 880 yd. apart, observe the angles of elevation to be 40° and 55°, respectively?



SUGGESTION. — Form two equations in two unknown numbers.

22. If $\triangle ABC$ is any triangle, show that the altitude $CD = b \sin A = a \sin B$.



23. Show from the results of Ex. 22 that

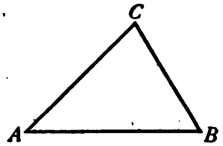
$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

24. By drawing another altitude in $\triangle ABC$, Ex. 22, show that $b \sin C = c \sin B$, and thus show that in any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This important relation is known as the *Law of Sines.*

25. In $\triangle ABC$, if $AB = 148$, $\angle A = 46°$, and $\angle B = 58°$, find AC.

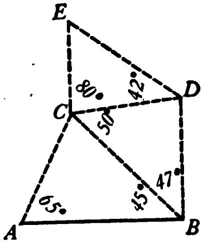


SUGGESTION. — $\angle C = 76°$. Why? Hence by the *law of sines* in Ex. 24,

$$\frac{AC}{\sin 58°} = \frac{148}{\sin 76°}.$$

Now substitute values of sin 58° and sin 76°, and solve for AC.

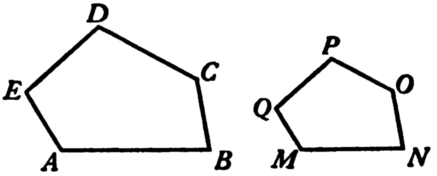
26. In surveying through great distances and over rough or mountainous country, a process called *triangulation* is employed. First a base line AB is carefully measured. Then a third point C is selected and $\angle BAC$ and $\angle ABC$ measured. By the *law of sines* CB is then computed. Another point D is then selected, and $\angle BCD$ and $\angle CBD$ measured. Then, by again applying the law of sines, CD is computed. Continuing, point E is selected, and $\angle ECD$ and $\angle CDE$ measured. From these measurements, ED is computed. In order to find ED it is only necessary to measure the one line AB. The process may be continued for any number of steps.



If $AB = 2180$ ft., and the angles measured are as indicated in the figure, find ED.

NOTE. — Let the students measure by triangulation the distance between two points by measuring a distant base line such as AB in Exercise 26. For measuring the angles, use may be made of some instrument such as that described in Exercise 16, page 40.

136. Theorem. — *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Hypothesis. $ABCD\cdots$ and $MNOP\cdots$ are similar polygons, with AB and MN as corresponding sides.

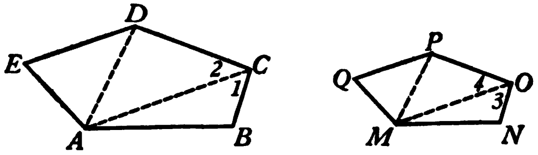
Conclusion. $\dfrac{AB+BC+CD+\text{etc.}}{MN+NO+OP+\text{etc.}}=\dfrac{AB}{MN}=\text{etc.}$

Proof. 1. $ABCD\cdots$ and $MNOP\cdots$ are similar, with AB and MN corresponding sides. Hyp.

2. $\therefore \dfrac{AB}{MN}=\dfrac{BC}{NO}=\dfrac{CD}{OP}=\text{etc.}$ Def. sim. poly.

3. $\therefore \dfrac{AB+BC+CD+\text{etc.}}{MN+NO+OP+\text{etc.}}=\dfrac{AB}{MN}=\text{etc.}$ § 118, (8)

137. Theorem. — *Two similar polygons can be divided into the same number of triangles, similar each to each and similarly placed.*



Hypothesis. $ABCD\cdots$ and $MNOP\cdots$ are similar polygons.

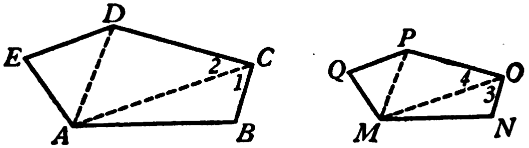
Conclusion. $ABCD\cdots$ and $MNOP\cdots$ can be divided into the same number of triangles, similar each to each and similarly placed.

Suggestions. Draw all diagonals possible from two corresponding vertices A and M. Prove $\triangle ABC \sim \triangle MNO$ by § 130. Then $\angle 1 = \angle 3$. Why? Next, prove $\angle 2 = \angle 4$. Also $\frac{AC}{MO} = \frac{BC}{NO}$. Why? Hence $\frac{AC}{MO} = \frac{CD}{OP}$. Why? Why, then, is $\triangle ACD \sim \triangle MOP$?

Write the proof in full.

Give the proof by drawing all possible diagonals from the corresponding vertices E and Q, instead of the diagonals from A and M.

138. Theorem. — *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



Hypothesis. Polygons $ABCD \cdots$ and $MNOP \cdots$ are composed of ⊿s ABC, ACD, etc., similar respectively to ⊿s MNO, MOP, etc., and similarly placed.

Conclusion. $ABCD \cdots \sim MNOP \cdots$.

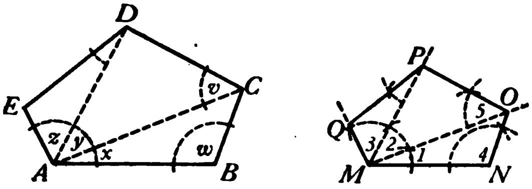
Suggestions. $\angle B = \angle N$. Why? Similarly, $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$. Hence $\angle C = \angle O$. Why? Similarly, prove $\angle D = \angle P$, etc.

$\frac{AB}{MN} = \frac{BC}{NO}$. Why? $\frac{BC}{NO} = \frac{AC}{MO} = \frac{DC}{PO}$. Why? Similarly $\frac{DC}{PO} = \frac{ED}{QP}$, etc.

Write the proof in full.

Give the proof by drawing diagonals from E and Q.

139. Construction. — *Upon a given line-segment corresponding to a side of a given polygon, construct a polygon similar to the given polygon.*



Given polygon $ABCD \cdots$ and line-segment MN.

Required to construct on MN as a side corresponding to AB, a polygon similar to $ABCD \cdots$.

Construction. 1. Draw all possible diagonals from A.

2. At M construct $\angle 1 = \angle x$, $\angle 2 = \angle y$, etc.

3. At N construct $\angle 4 = \angle w$, giving point O.

4. At O construct $\angle 5 = \angle v$, giving point P, etc.

5. Then polygon $MNOP \cdots \sim$ polygon $ABCD \cdots$.

Proof. The proof is left to the student.

EXERCISES

1. The perimeters of two similar polygons are 144 yd. and 256 yd., respectively. A side of the first is 18 yd. Find the corresponding side of the second.

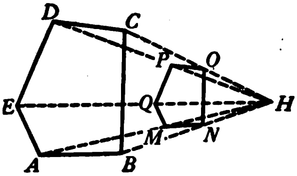
2. The sides of a polygon are 8 in., 12 in., 15 in., 6 in., and 20 in. The side of a similar polygon corresponding to the 8-in. side is 6 in. Find the perimeter of the similar polygon.

3. The perimeter of a polygon is p and one side is x. If the perimeter of a similar polygon is q, find the side corresponding to x.

4. A rectangular field is w yd. wide and l yd. long. Find the perimeter of a similar field $3w$ yd. wide.

5. Prove that two corresponding diagonals of two similar polygons have the same ratio as a pair of corresponding sides.

6. Prove that if two polygons are each similar to a third polygon, they are similar to each other.

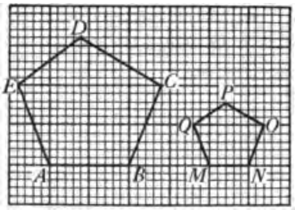


7. The vertices of a given polygon *ABCDE* are joined to a given point *H*. A line-segment *MN* is drawn parallel to *AB* so that *M* is on *AH* and *N* on *BH*. Then $NO \parallel BC$, $OP \parallel CD$, $PQ \parallel DE$, and *QM* are drawn as in the figure. Prove $MNOPQ \sim ABCDE$.

SUGGESTION. — How may *QM* be proved parallel to *EA*? Derive proportions to show that $\frac{EH}{QH} = \frac{AH}{MH}$.

8. Polygon *ABCDE* is formed by joining points of intersection of lines on squared paper. Polygon *MNOPQ* is formed by joining corresponding intersections of lines separated by only one half the interval.

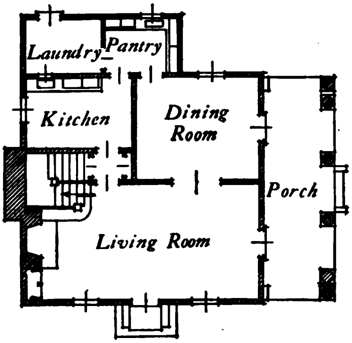
Prove that $ABCDE \sim MNOPQ$.



9. Explain how squared paper may be used to construct similar polygons the ratio of whose sides is 3; the ratio of whose sides is 5.

10. Draw any quadrilateral. Upon a given line-segment as a side corresponding to a given side of the quadrilateral, construct by use of compasses and straightedge a similar quadrilateral.

140. Maps and plans. — A **map** or **plan** is a figure similar to the figure formed by the object which it represents. Thus, a map of a state is a drawing similar to the figure formed by the state itself. The drawing in the margin shows an architect's *floor plan* of a house.



A map or plan is always *drawn to scale*, i.e., in the map or plan the distances are made *proportional* to the actual distances which they represent.

Thus, a map of the United States which is drawn so that 200 miles measured anywhere across the country is represented on the map by a distance of 1 inch, is said to be *drawn to the scale of* 200 *miles to an inch* (scale: 200 mi. = 1 in.).

EXERCISES

1. Consult a map of the United States. Find the scale to which it is drawn. Find from this map the number of miles in a straight line from Boston to San Francisco.

2. A map of Illinois drawn to the scale of 200 mi. to an inch is $1\frac{5}{6}$ in. long. How many miles long is the state?

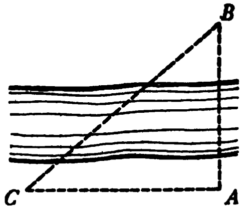
3. On a map drawn to the scale of 240 mi. to an inch, the distance from Chicago to Denver is $3\frac{9}{16}$ in. How many miles is it from Chicago to Denver?

4. In the house of which the floor plan is shown in § 140, the width of the living room is 15 ft. By measuring the width of the living room in the plan, find the scale to which the plan is drawn.

5. From the scale found in Ex. 4, determine the number of feet in the width of the porch. Find the length of the porch.

6. How many feet wide is the dining room of this house?

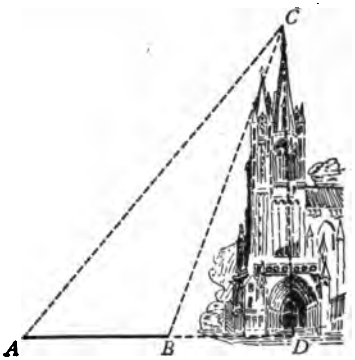
7. Draw a rectangle representing a rectangular field that is 1200 ft. long and 480 ft. wide to a scale of 240 ft. to an inch. What are the dimensions of the drawing?



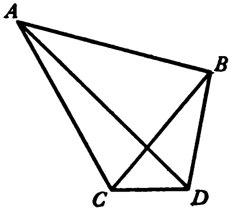
8. The distance from A to the inaccessible point B may be obtained as follows: Measure a base line AC. Measure $\angle ACB$ and $\angle BAC$. Then construct a *map* of the measurements to scale, and determine from the map the distance from A to B.

If $AC = 960$ ft., $\angle ACB = 40°$, and $\angle BAC = 90°$, draw a map of the measurements to the scale of 160 ft. to an inch, and compute AB from the map.

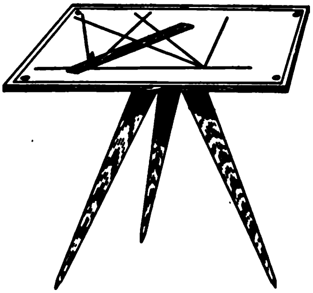
9. In order to find the height of a church spire *CD*, the base line *AB* is measured 75 ft. long toward the foot of the spire *D*. It is found that $\angle DAC = 50°$ and $\angle DBC = 80°$. Make a drawing of these measurements to the scale of 25 ft. to an inch, and compute the height *CD* of the spire.



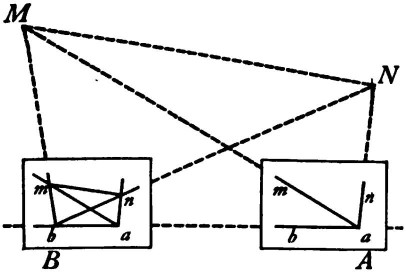
10. *A* and *B* are two forts in the lines of the enemy, and it is desired to know their distance apart and their distances from our lines. From point *C* in our lines, we measure $\angle ACD$ and $\angle BCD$. Then we go to a second point *D* and measure $\angle CDA$ and $\angle CDB$. $\angle ACD = 120°$, $\angle BCD = 50°$, $\angle CDA = 45°$, $\angle CDB = 100°$, and $CD = 2000$ ft. Draw a plan to the scale of 500 ft. to the inch, and find the distances *AC*, *BD*, and *AB*.



141. The plane table. — A **plane table** consists of a drawing board mounted upon a tripod. A sheet of paper is pinned on the board, and a straightedge is laid on it for sighting and drawing lines. The instrument is used for finding the distances between inaccessible points and for making maps of small areas.



In using the plane table for finding the distance between two points *M* and *N*, the instrument is set up at any convenient point *A*. A sheet of paper is fastened on the board, and a pin stuck through the paper into the board at a point *a* directly over *A*. The straightedge is placed against the pin, and a base line *ab* drawn on the paper toward a second point *B*. Lines *an* and *am* are then drawn toward *N* and *M*, respectively. The plane table is then removed and set up over point *B*, so that the line *ab* on the paper is directly above *B* and points directly back to the old station *A*. The pin is then removed to a point *b* directly over *B*, in line *ab*. Lines *bn* and *bm* are then drawn toward *N* and *M*, respectively. The distance *AB* is measured. Lines *an* and *bn* meet at *n*, and *am* and *bm* meet at *m*. Then *mn* is drawn and measured. Also *ab* is measured. From these measurements the distance *MN* is computed by proportion.



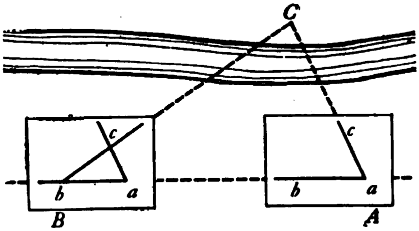
In using the plane table for making a map or for other kinds of measurements, the procedure is very similar to that described above.

EXERCISES

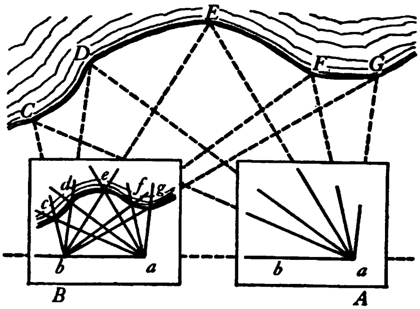
1. Prove that the line *MN* in the figure of § 141 may be found from the proportion $\frac{MN}{AB}=\frac{mn}{ab}$.

SUGGESTION. — For proving this proportion use the proportion in similar triangles and § 130.

2. Study the figure in the margin, and explain in their natural order the steps in finding the distance from a point *A* to a point *C* by use of the plane table.



3. The following drawing shows the method of making a map *cdefg* of the river bank *CDEFG* by use of the plane table. Study the drawing, and explain how the points *c*, *d*, *e*, *f*, and *g* of the map are obtained as the intersections of corresponding lines drawn to the points *C*, *D*, *E*, *F*, and *G* from two points in the base line.



If $AB = 640$ ft., $ab = 16$ in., and $ad = 20$ in., how far is it from *A* to *D*?

If *cf* is measured and found to be 10 in., how far is it from *C* to *F*?

Note. — It will be found a very interesting and valuable experience, if the school will provide, or the students make, a plane table and use it out of doors in doing such work as is suggested in the exercises above.

CHAPTER VII

INEQUALITIES: METHODS OF ATTACK

142. Inequalities. — An **inequality** is a statement that two numbers or magnitudes are unequal.

The symbols of inequality are $>$, meaning "is greater than," and $<$, meaning "is less than."

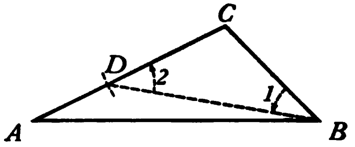
Thus "a is greater than b" is written $a > b$.

And "a is less than b" is written $a < b$.

The inequalities $a > b$ and $x > y$ are of the *same order.*

The inequalities $a > b$ and $x < y$ are of the *reverse order.*

143. Theorem. — *If two sides of a triangle are unequal, the angles opposite these sides are unequal, the angle opposite the greater side being the greater.*



Hypothesis. In $\triangle ABC$, $AC > BC$.

Conclusion. $\angle B > \angle A$.

Proof. 1. $AC > BC$. Hyp.

2. Then let CD be a part of AC equal to BC. Draw DB.

3. Then $\angle B > \angle 1$. Ax. X

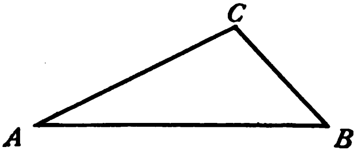
4. But $\angle 2 = \angle 1$. § 70

5. $\therefore \angle B > \angle 2$. Ax. XII

6. $\angle 2 > \angle A$. § 51

7. $\therefore \angle B > \angle A$. Ax. XI

144. Theorem. — *If two angles of a triangle are unequal, the sides opposite these angles are unequal, the side opposite the greater angle being the greater.*



Hypothesis. In $\triangle ABC$, $\angle B > \angle A$.

Conclusion. $AC > BC$.

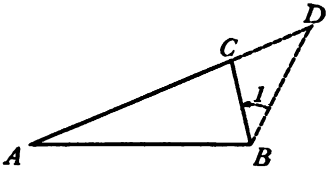
Suggestions. Use indirect proof. AC is either greater than, equal to, or less than BC. Show by § 70 that AC cannot equal BC. Show by § 143 that AC cannot be less than BC.

Write the proof in full.

145. Corollary. — *The perpendicular from a point to a line is the shortest line-segment that can be drawn from the point to the line.*

The proof is left to the student.

146. Theorem. — *The sum of any two sides of a triangle is greater than the third side.*

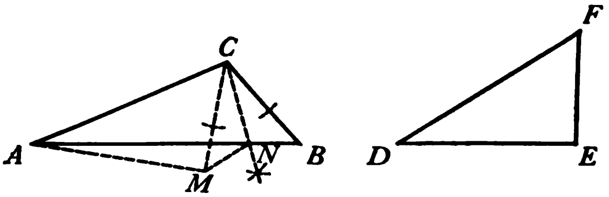


Hypothesis. $\triangle ABC$ is any triangle.

Conclusion. $AC + BC > AB$.

Suggestions. Produce AC through C to D, making $CD = BC$. Compare $\angle 1$ and $\angle D$, $\angle DBA$ and $\angle 1$, then $\angle DBA$ and $\angle D$. Why, then, is $AC + CD > AB$? Why, then, is $AC + BC > AB$?

147. Theorem. — *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the sides opposite these angles are unequal, the greater side being opposite the greater angle.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ have $AC = DF$, $BC = EF$, and $\angle ACB > \angle DFE$.

Conclusion. $AB > DE$.

Proof. 1. $AC = DF$, $BC = EF$, $\angle ACB > \angle DFE$. Hyp.

2. $\therefore \triangle DEF$ may be placed upon $\triangle ABC$ so that DF coincides with AC, taking the position of $\triangle AMC$. § 10

3. $\angle ACB > \angle ACM$ and $MC = BC$. Ax. XII

4. $\therefore CM$ lies within $\angle ACB$. § 13 and Ax. X

5. Draw CN bisecting $\angle MCB$ and intersecting AB at N, and draw MN.

6. Then $\triangle MNC \cong \triangle BNC$. § 63

7. $\therefore MN = NB$. Def. congruence

8. $AN + MN > AM$. § 146

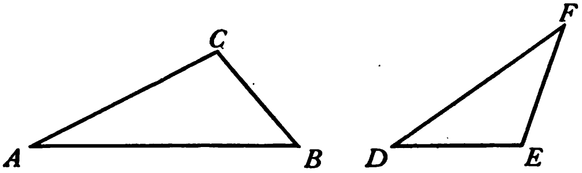
9. $\therefore AN + NB > AM$, or $AB > DE$. Ax. XII

In the special case when point M falls on AB, how may the proof be shortened?

Is it possible to draw a figure in which M falls within $\triangle ABC$?

Draw a figure in which the included angles ACB and DFE are both acute, and prove the theorem.

148. Theorem. — *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angles opposite these sides are unequal, the greater angle being opposite the greater side.*



Hypothesis. In $\triangle ABC$ and DEF, $AC = DF$, $BC = EF$, $AB > DE$.

Conclusion. $\angle C > \angle F$.

Suggestions. Use indirect proof. Either $\angle C > \angle F$, $\angle C = \angle F$, or $\angle C < \angle F$. Show by § 63 that $\angle C$ is not equal to $\angle F$. Show by § 147 that $\angle C$ is not less than $\angle F$.

Write the complete proof.

EXERCISES

State in which cases it is possible to draw triangles with sides of the following lengths:

1. 4 in., 5 in., 6 in.
2. 2 in., 4 in., 6 in.
3. 6 in., 12 in., 20 in.
4. $6\frac{1}{2}$ in., $7\frac{1}{2}$ in., 13 in.

5. The hypotenuse of a right triangle is greater than either leg.

6. If an isosceles triangle is obtuse, the base is the longest side.

7. In any triangle, any side is greater than the difference between the other two sides.

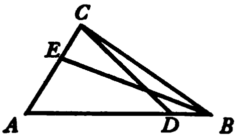
8. Any side of a polygon is less than the sum of all of the other sides.

SUGGESTION. — Draw all diagonals possible from one end of the given side.

9. The perimeter of any quadrilateral is greater than the sum of the diagonals.

10. The sum of the diagonals of any quadrilateral is greater than the sum of a pair of opposite sides.

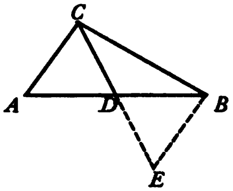
11. In $\triangle ABC$, $AB > AC$ and $DB = EC$. Prove $BE > CD$.



12. If P is any point between B and C in the base BC of an isosceles triangle ABC, then $AP < AB$.

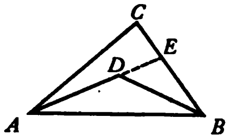
13. The median to one side of a triangle is less than one half of the sum of the other two sides.

SUGGESTIONS. — Produce CD through D to E so that $DE = CD$. Now it may be proved that $CD < \frac{1}{2}(AC + BC)$, if it is first proved that $2\,CD$ or $CE < AC + BC$.

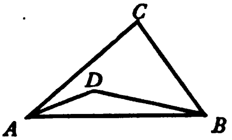


14. In any triangle the sum of two sides is greater than the sum of two line-segments drawn from a point within the triangle to the extremities of the third side.

SUGGESTIONS. — Produce AD to meet BC at E. Prove $AC + CB > AE + EB$. Similarly, prove $AE + EB > AD + DB$. Then apply Ax. XI.

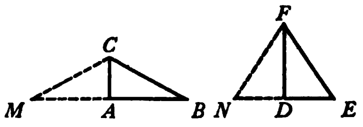


15. D is a point within $\triangle ABC$ such that $BD = BC$. Prove $AC > AD$.



16. If two right triangles have their hypotenuses equal and an acute angle of one greater than the corresponding acute angle of the other, the legs opposite these angles are unequal, the longer leg being opposite the larger angle.

SUGGESTIONS. — If $BC = EF$ and $\angle C > \angle F$, produce AB through A to M, making $AM = AB$, and draw CM; also produce DE through D to N, making $DN = DE$, and draw FN. Now show that $MB > NE$, and hence $AB > DE$.

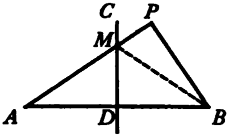


17. If two sides of a triangle are unequal, the median to the third side makes the larger angle with the shorter of the unequal sides.

Suggestions. — Draw a triangle ABC with $AC > CB$; let CD be the median. Prove that $\angle DCB > \angle ACD$. Produce CD to E, making $CD = DE$; draw AE. Compare $\angle DCB$ with $\angle E$, and $\angle ACD$ with $\angle E$.

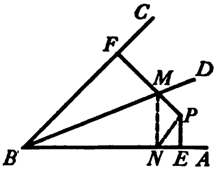
18. Using the figure drawn for Ex. 17, prove that $CD < \frac{1}{2}(AC + CB)$.

19. Give a direct proof that any point not on the perpendicular bisector of a line-segment is not equidistant from the ends of the segment.



Suggestion. — If CD is the perpendicular bisector of AB and P any point not on CD, let P be on the same side of CD as B, and let AP intersect CD at M. Draw MB. Prove $PB < PA$. PB is less than the sum of what two line-segments?

20. Give a direct proof that a point within an angle but not on the bisector of the angle is not equidistant from the sides.



Suggestion. — If BD bisects $\angle ABC$, and P is within $\angle ABC$ but not on BD, let P be between AB and BD. Prove $PE < PF$. Draw $MN \perp BA$ and draw PN. Show that $PE < PN$, $PN < PM + MN$, etc.

149. Method of attack. — The student's experience in proving the unproved theorems and corollaries and the exercises in the preceding pages should have revealed the fact that there is a definite procedure or *method of attack* in such proofs.

Success in demonstrations requires, first of all, thorough familiarity of the student with all preceding theorems, corollaries, and definitions upon which the proofs may depend. Most proofs involve a comparison of line-segments, angles, or ratios. *It would be well for the student at this point to review all of the theorems, corollaries, and definitions, and to fix them thoroughly in mind as providing means of proving one or other of the following fundamental relations:*

Equality of line-segments.

Thus, line-segments are proved equal by the theorems on congruence of triangles, on parallelograms, on isosceles triangles, etc.

Inequality of line-segments.

Thus, line-segments are proved unequal by the theorems on unequal angles of a triangle, etc.

Equality of angles.

Angles are proved equal by showing that they are complements or supplements of equal angles, or vertical angles, or by the theorems on congruence of triangles, on isosceles triangles, on transversals of parallel lines, etc.

Inequality of angles.

Angles are proved unequal by the relations between acute, right, and obtuse angles, by theorems on unequal sides of a triangle, etc.

Parallel lines, perpendicular lines, proportions, etc.

Directions

In attempting to *discover and give a direct proof* of a theorem, corollary, or exercise in the nature of a theorem, the following directions should be followed:

(1) *Draw an accurate but general figure.*

The figures should be formed by accurate geometric construction when possible. An inaccurately drawn figure interferes with clear thinking.

Do not draw an isosceles or right triangle unless such is specifically demanded.

(2) *Get clearly in mind the details of the hypothesis and conclusion, being sure of the exact meanings of all terms involved.*

(3) *If the hypothesis and conclusion do not suggest theorems, etc., related to both, so that the steps of the proof are seen at once, proceed to discover the proof as follows:*

Recall theorems, corollaries, definitions, related to the thing to be proved until one is chosen from which the conclusion would follow. See if, in the light of the hypothesis, such theorem, corollary, or definition applies at once to the figure. If not, try to discover by the same process a second theorem, etc., from which this one would follow. Continue such steps, if possible, until, in the light of the hypothesis, some known truth is seen to apply to the figure.

If successful, begin the proof by applying this known truth to the figure.

The process suggested above of discovering with what step to begin a proof is called **analysis.**

The steps of an analysis may be expressed symbolically as follows: If A is to be proved, reason thus: A will follow if B is first proved; B will follow if C is first proved; C will follow if D is first proved; but D is true by hypothesis; hence begin by proving C.

(4) *If no known truth related to the conclusion can be found that applies to the figure as given, try to draw one or more auxiliary lines that will make the figure into one such that some theorem, corollary, or definition will apply.*

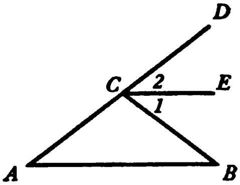
If no *direct proof* can be found, seek to use *indirect proof,* by assuming the conclusion to be false and showing that this assumption leads to a contradiction of a known truth.

The following miscellaneous exercises illustrate the method of attack. Study very carefully the analysis given in each of the first few exercises of the list, and endeavor to see how the general directions which are given above are carried out in each of these exercises.

MISCELLANEOUS EXERCISES FOR REVIEW

1. If the bisector of an exterior angle at a vertex of a triangle is parallel to the opposite side, the triangle is isosceles.

Hypothesis. — CE bisects $\angle BCD$, an exterior angle of $\triangle ABC$, and $CE \parallel AB$.



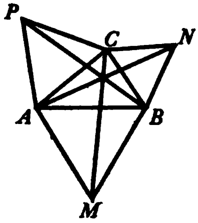
Conclusion. — $\triangle ABC$ is isosceles.

Suggestions. — The exercise may be *analyzed* thus:

$\triangle ABC$ is isosceles by § 74 if $\angle A = \angle B$. It may be proved by Ax. I that $\angle A = \angle B$ if it is first shown that $\angle A = \angle 2$, $\angle B = \angle 1$, and $\angle 1 = \angle 2$. These three equalities follow directly from facts of the hypothesis. Hence, begin the proof by showing that $\angle A = \angle 2$, $\angle B = \angle 1$, and $\angle 1 = \angle 2$.

Write the proof in full. Do not include any of the analysis in the proof. It is only a means of finding where to begin the proof.

2. If $\triangle AMB$, $\triangle BNC$, and $\triangle APC$ are equilateral triangles drawn on the sides of $\triangle ABC$, AN, BP, and CM are equal.

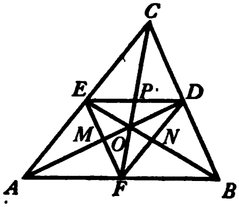


Suggestions. — The exercise may be *analyzed* thus:

$AN = BP$ if it is proved that $\triangle ACN \cong \triangle PCB$. Since $PC = AC$ and $CN = CB$ by definition of an equilateral triangle, $\triangle ACN \cong \triangle PCB$ by § 63 if it can be proved that $\angle ACN = \angle PCB$. And $\angle ACN = \angle PCB$ if $\angle PCA = \angle BCN$. But $\angle PCA = \angle BCN$ since each is 60° by § 72. Hence begin the proof by showing that $\angle PCA = \angle BCN$.

Write the proof in full.

3. If AD, BE, and CF are the medians of $\triangle ABC$, and intersect at O, then O is the intersection of the medians of $\triangle DEF$ also.

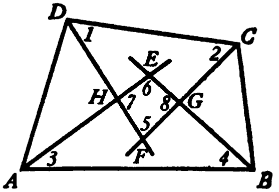


Suggestions. — The exercise may be *analyzed* thus:

O is the intersection of the medians of $\triangle DEF$ if $EP = PD$, $FM = ME$, and $FN = ND$. Now $EP = PD$ if $CEFD$ is a parallelogram, and $CEFD$ is a parallelogram if $FD \parallel AC$ and

$FD = \frac{1}{2} AC$, which is true by § 97. Hence, begin the proof by showing that $FD \parallel AC$ and $FD = \frac{1}{2} AC$.

4. If the bisectors of the angles of quadrilateral $ABCD$ form a quadrilateral $EHFG$, the opposite angles of the quadrilateral $EHFG$ are supplementary.



SUGGESTIONS. — The exercise may be *analyzed* thus:

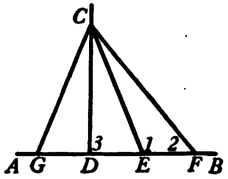
Since the sum of the angles of $\triangle AEB$ and $\triangle DFC$ is 2 st. $\angle$s, $\angle 5 + \angle 6 =$ st. $\angle$ if $\angle 1 + \angle 2 + \angle 3 + \angle 4 =$ st. $\angle$. Now $\angle 1 + \angle 2 + \angle 3 + \angle 4 =$ st. $\angle$, if $\angle A + \angle B + \angle C + \angle D = 2$ st. $\angle$s. But the latter follows from the hypothesis by § 101. Hence, begin by proving that $\angle A + \angle B + \angle C + \angle D = 2$ st. $\angle$s.

5. The quadrilateral formed by the bisectors of the angles of a parallelogram is a rectangle.

6. The quadrilateral formed by the bisectors of the angles of a rectangle is a square.

7. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

8. If two oblique line-segments are drawn to a line from a point in a perpendicular to the line, the segment meeting the line at the greater distance from the foot of the perpendicular is the greater. That is, if $CD \perp AB$, and $DF > DE$, then $CF > CE$.



SUGGESTIONS. — First, suppose CE and CF on the same side of CD. Then $CF > CE$ if $\angle 1 > \angle 2$. Now $\angle 1 > \angle 2$ if $\angle 1 > \angle 3$ and $\angle 3 > \angle 2$, etc. Second, suppose the oblique segments on opposite sides of CD, as CF and CG. In that case, $CF > CG$ if CE is taken on the same side of CD as CF so that $DE = DG$, and CF proved greater than CE and $CE = CG$.

9. Prove the converse of Ex. 8; that is, if $CF > CE$, then $DF > DE$.

10. The triangle formed by joining the middle points of the sides of an isosceles triangle is isosceles.

11. If two medians of a triangle are equal, the triangle is isosceles.

12. The line-segments which join the middle points of the adjacent sides of an isosceles trapezoid form a rhombus.

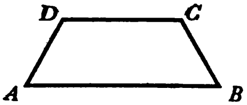
13. The triangle formed by the line-segments connecting the middle points of the three sides of a given triangle is similar to the given triangle.

14. If the middle point of one side of a triangle is equidistant from the three vertices, the angle opposite that side is a right angle.

15. If a transversal cuts two parallel lines, the bisectors of the four interior angles form a rectangle.

16. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

17. In quadrilateral $ABCD$, if $AD = BC$ and $\angle D = \angle C$, then $DC \parallel AB$.



18. The sum of the line-segments drawn from any point within a triangle to the three vertices is less than the sum of the three sides.

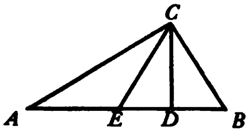
19. The sum of the line-segments drawn from any point within a triangle to the three vertices is greater than one half of the sum of the three sides.

20. The sum of the distances of any point within any polygon from all of the vertices is greater than one half of the perimeter.

21. Any side of a triangle is less than one half of the perimeter.

22. If two angles of a triangle are unequal, the bisector of the other angle is not perpendicular to the opposite side.

23. In the right triangle ABC, AB is the hypotenuse, $CD \perp AB$, and CE is the median. Prove that $\angle DCE = \angle B - \angle A$.



24. If the exterior angles at A and B of $\triangle ABC$ are bisected by AD and BD, respectively, then $\angle ADB = 90° - \frac{1}{2}\angle C$.

25. The parallelogram formed by joining the middle points of the adjacent sides of a quadrilateral has the intersection of its diagonals at the middle point of the line-segment joining the middle points of the diagonals of the quadrilateral.

26. In quadrilateral $ABCD$, AB is the longest side and CD the shortest. Prove $\angle C > \angle A$ and $\angle D > \angle B$.

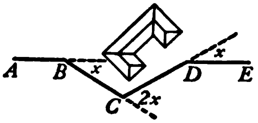
27. If lines are drawn bisecting the angles of any parallelogram,

(1) each bisector cuts off an isosceles triangle from the parallelogram;

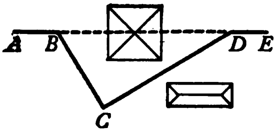
(2) the segments of the bisectors of two adjacent angles included by the sides of the parallelogram bisect each other;

(3) the diagonals of the rectangle formed by the bisectors are parallel to the sides of the parallelogram.

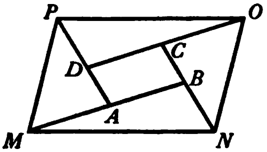
28. A surveyor extends a straight line AB beyond an obstacle as follows: At B he turns off any convenient angle x, and runs a line to C. At C he turns off an angle equal to $2x$, as in the drawing, and runs a line to D, making $CD = BC$. At D he turns off an angle equal to x, giving the line DE. Prove that AB and DE are in one straight line.



29. When other methods cannot be conveniently used, a surveyor extends a straight line AB beyond an obstacle as follows: He turns off a convenient angle at B and runs a line to C. Then he runs CD at right angles to BC. If DE is the prolongation of AB, CD is computed by a trigonometric ratio. If the angle turned off at B is 58° and $BC = 360$ ft., how long must CD be taken?



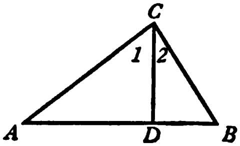
30. The sides of $\square ABCD$ are produced in succession to points M, N, O, and P, so that $DP = BN$ and $AM = CO$. Prove that $MNOP$ is a parallelogram.



31. If two triangles have their sides respectively parallel, they are similar.

32. If two triangles have their sides respectively perpendicular, they are similar.

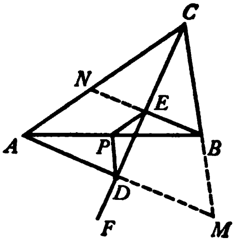
33. The perpendicular from a vertex of any triangle to the opposite side divides the angle at the vertex into two angles whose difference equals the difference between the other two angles of the triangle. ($\angle 1 - \angle 2 = \angle B - \angle A$.)



34. In $\triangle ABC$, CF bisects $\angle C$. AD and BE are drawn perpendicular to CF, and P is the middle point of AB. Prove that

$$PD = PE = \tfrac{1}{2}(AC - BC).$$

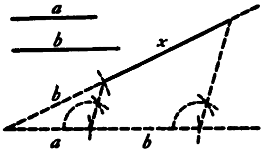
Suggestion. — Produce CB, and let AD produced meet it at M. Also produce BE to meet AC at N.



35. In the figure of Ex. 34, prove that

$$\angle PAD = \angle PBE = \tfrac{1}{2}(\angle B - \angle A).$$

36. In the mean proportion $\frac{a}{b} = \frac{b}{x}$, the term x is called the *third proportional* to a and b. Show from the figure that the third proportional to a and b is found by the same construction as the fourth proportional to three line-segments, except that b is used twice.

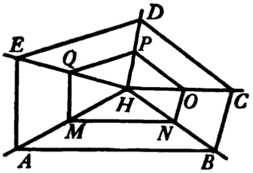


37. If a, b, and c are given line-segments, show how to construct $\frac{bc}{a}$.

Suggestion. — Let $x = \frac{bc}{a}$. Then, solving, $\frac{a}{b} = \frac{c}{x}$.

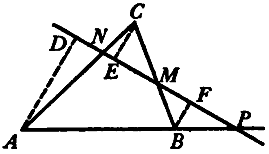
38. If a and b are given line-segments, construct $\frac{b^2}{a}$.

39. Lines HA, HB, HC, HD, and HE radiate from a point H. Polygon $ABCDE$ and polygon $MNOPQ$ are drawn with their vertices on these lines. $MN \parallel AB$, $NO \parallel BC$, $OP \parallel CD$, $PQ \parallel DE$, and Q is joined to M. Prove $QM \parallel EA$. Prove $MNOPQ \sim ABCDE$.

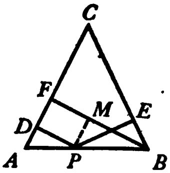


40. In $\triangle ABC$, line MN intersects BC at M, AC at N, and AB produced at P. Prove $AP \times BM \times CN = PB \times MC \times NA$.

Analysis. — This may be proved if proper proportions are obtained containing $\frac{AP}{PB}$, $\frac{BM}{MC}$, and $\frac{CN}{NA}$, and the ratios multiplied. For obtaining these proportions, draw AD, CE, and BF perpendicular to MN.

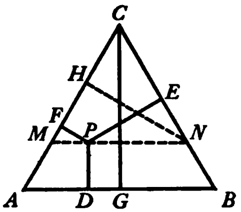


Note. — This is known as the *Theorem of Menelaus*.



41. If from any point P in the base of an isosceles triangle perpendiculars are drawn to the equal sides, the sum of their lengths is constant for all positions of the point and is equal to the perpendicular from an extremity of the base to the opposite side. That is, $PD + PE = BF$.

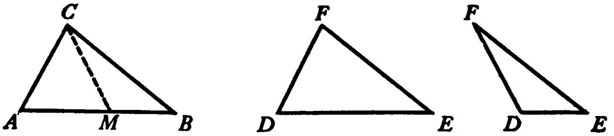
SUGGESTION. — Draw $PM \parallel AC$, meeting BF at M. It may be proved that $PD + PE = BF$ if it is first proved that $PD = MF$ and $PE = BM$.



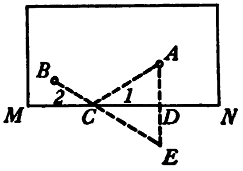
42. From any point within an equilateral triangle the sum of the perpendiculars to the three sides is equal to an altitude.

SUGGESTION. — Draw $MN \parallel AB$ through P. Seek to apply Ex. 41 to the figure.

43. If two triangles have two sides of one equal respectively to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are equal or supplementary, and if equal the triangles are congruent.



SUGGESTION. — If $AC = DF$, $BC = EF$, and $\angle B = \angle E$, then $\triangle DEF$ may be placed upon $\triangle ABC$ so that EF shall coincide with BC and $\angle A$ and $\angle D$ shall be on the same side of BC. Then why will DE fall along AB? Then D will either fall upon A, in which case $\triangle ABC \cong \triangle DEF$, or D will fall at some other point M of AB. In the latter case prove $\angle BMC$ and $\angle A$ supplementary.

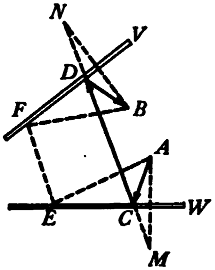


44. When a billiard ball strikes a side of the table, the path along which it rebounds makes an angle with the side of the table equal to that made by the path along which it strikes.

A ball at A is to be driven against the side MN so that when it rebounds it will strike a given ball at B. Show that the following geometric construction will locate the point C at which the ball must strike the side MN of the table: Draw $AD \perp MN$ and produce it to E so that $DE = AD$, and draw EB intersecting MN at C. (Prove $\angle 1 = \angle 2$.)

45. Prove that the path ACB is the shortest path by which the billiard ball in Ex. 44 could travel from A to B and strike the side MN of the table.

Suggestion. — Let the ball strike MN at any other point P, and draw AP, PB, and PE. Prove $AC + CB < AP + PB$.

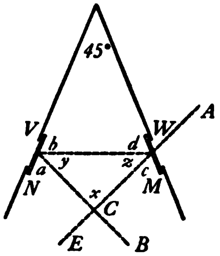


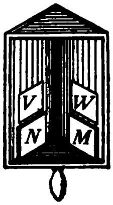
46. Light which strikes a mirror is reflected at an angle with the mirror equal to the angle at which it strikes it.

Two mirrors, W and V, are placed at an angle. Light from a point A travels to the point B along the path $ACDB$. Prove that $ACDB$ is the shortest path that light can travel from A to B and be reflected from both mirrors.

Suggestion. — Let $AEFB$ be any other path. Prove $ACDB < AEFB$. Observe that $ACDB = MN$.

47. The *optical square* is an instrument used in field work, such as forestry, for laying off right angles. It consists of a box with triangular top and bottom, and only two side walls, which are set at an angle of exactly 45° with each other. In these walls are cut openings, or windows, W and V. Below the windows, mirrors, M and N, are fastened against the walls. The observer, whose eye is at E (See diagram), looks into the box through the open side, and holds it so that an object A can be seen through window W. At the same time the image of an object B is seen in mirror M, and in line with A. The principle is that light



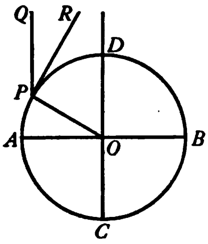


from B strikes mirror N, is reflected to mirror M, then reflected to the eye at E. By the law of reflection of light, $\angle a = \angle b$ and $\angle c = \angle d$. Show that $\angle BCA$ is a right angle.

SUGGESTION. — Form equations $\angle a = \angle b$, $\angle c = \angle d$, $\angle a + \angle b + \angle y = 180°$, $\angle c + \angle d + \angle z = 180°$, $\angle b + \angle d + 45° = 180°$, and $\angle x + \angle y + \angle z = 180°$. Show how each of these equations is obtained. Solve for $\angle x$.

NOTE. — An optical square can easily be made by the student. It may be used in a large number of out-of-door problems.

48. In the diagram, the circle represents a section of the earth through its center. PQ is the line of sight to the North Star from a point P of the earth's surface. PR is a horizontal line at P, and $\angle RPQ$ represents the *altitude*, or angle of elevation above the horizon, of the North Star. The horizontal line PR is perpendicular to the vertical line OP at P. CD is the axis of the earth, and also points approximately to the North Star. Because of the great distance of the star, CD may be considered parallel to PQ. AB is a diameter of the equator, and is perpendicular to CD. The number of degrees in $\angle POA$ measures the *latitude* of P.



Prove that the *latitude* of P equals the *altitude* of the North Star as observed from P, that is, $\angle POA = \angle RPQ$.

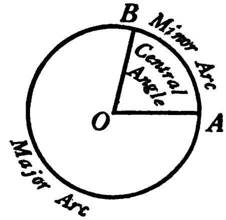
NOTE. — If an instrument were made as suggested in the note, page 20, the student could find the latitude of the school by measuring the altitude of the North Star.

CHAPTER VIII

CIRCLES

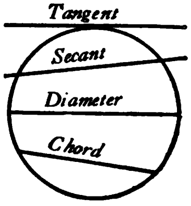
150. Circles. — The definition of a **circle** was given in § 6, also the definitions of **radius, diameter,** and **arc** of a circle.

It was shown in § 103 that *a circle is a locus of points at a given distance from a given point.*



A circle is divided into two arcs by any two of its points, as A and B. If these two arcs are equal, each is called a **semicircle.** If they are unequal, the smaller is called the **minor arc** and the larger is called the **major arc.** Unless otherwise specified, *arc AB* (written $\widehat{AB}$) shall be taken to mean the *minor arc* terminating at A and B.

An angle formed by two radii is called a **central angle,** as $\angle AOB$.



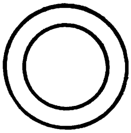
An angle is said to *intercept* the arc cut off by its sides, and the arc is said to *subtend* the angle.

A straight line which *touches* a circle at only one point is called a **tangent.** The point is called the **point of contact.**

A straight line which intersects a circle at two points is called a **secant.**

A line-segment whose ends are on a circle is called a **chord.** It is evident that a *diameter* is a chord which passes through the center. A chord is said to *subtend* the arcs into which its extremities cut the circle.

Two circles which have the same center are called **concentric**.



CONCENTRIC CIRCLES

151. Fundamental properties of circles. — The following are fundamental properties of circles :

(1) *A diameter of a circle equals two times a radius.*

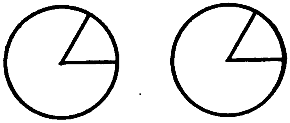
For, a diameter is composed of two radii by definition.

(2) *Radii of the same circle or equal circles are equal.*

Radii of the same circle are equal by definition of a circle. And since equal circles may be made to coincide, their radii may be made to coincide and hence are equal.

(3) *If the radii of two circles are equal, the circles are equal.*

For, by making the equal radii coincide, the circles coincide.

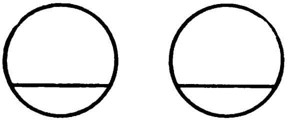


(4) *In the same circle or equal circles, equal central angles intercept equal arcs.*

For, when the equal angles are superposed so that they coincide, the intercepted arcs coincide.

(5) *In the same circle or equal circles, equal arcs subtend equal central angles.*

For, when the equal arcs are superposed so that they coincide, the central angles coincide.



(6) *In the same circle or equal circles, equal arcs are subtended by equal chords.*

For, when the equal arcs are superposed so that they coincide, the chords which subtend the arcs coincide.

(7) *In the same circle or equal circles, equal chords subtend equal arcs.*

For, when the equal chords are superposed so that they coincide, the subtended arcs coincide.

(8) *A point is at a distance from the center of a circle equal to, greater than, or less than the radius, according as it is on, outside of, or within the circle; and conversely.*

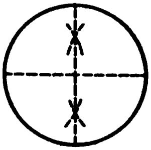
This follows readily from the definition of a circle and Axiom X.

These fundamental properties of circles should be thoroughly thought out and understood, as well as memorized, by the student. They are very important principles, and will be used much in the subsequent work. Yet, they are so simple that more formal proofs of them than those suggested above are not demanded.

EXERCISES

1. Divide a given circle into four equal arcs.

Suggestion. — Draw two perpendicular diameters. Why are the central angles equal? Why then are the arcs equal?



2. Divide a given circle into eight equal arcs.

Suggestion. — Draw two perpendicular diameters, then draw the bisectors of the four angles at the center.

3. Explain how to divide a given circle into sixteen equal arcs. Into thirty-two equal arcs. Into sixty-four equal arcs, etc.

Into how many equal arcs, in general, may a circle be divided by this method?

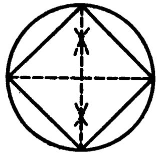
4. Divide a given circle into six equal arcs.

Suggestion. — Construct an equilateral triangle upon any radius as a side. How many degrees are there in the central angle? How may the construction be completed?

5. Divide a given circle into three equal arcs.

6. Explain how to divide a given circle into twelve equal arcs. Into twenty-four equal arcs, etc.

Into how many equal arcs, in general, may a circle be divided by this method?



7. Construct an equilateral quadrilateral with its vertices on a given circle.

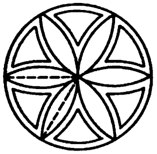
Suggestion. — First divide the circle into four equal arcs. Then draw the chords of these arcs. Why are the chords equal?

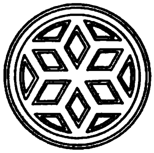
8. Construct an equilateral hexagon with its vertices on a given circle.

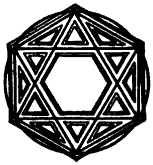
9. Construct an equilateral octagon with its vertices on a given circle.

10. Construct an equilateral triangle with its vertices on a given circle.

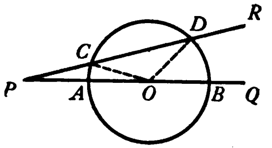
11. These designs are made by dividing a circle into equal arcs. Study and explain their constructions.







12. Make an original design based upon the division of a circle into equal arcs.



13. PQ is drawn through the center O, intersecting the circle at A and B; and PR is any other secant intersecting the circle at C and D. Prove $PA < PC$, and $PB > PD$.

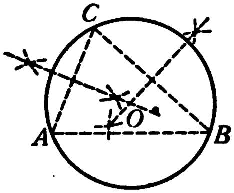
Suggestions. — For proving that $PA < PC$, first compare $PA + AO$ and $PC + CO$.

For proving that $PB > PD$, first compare PB and $PO + OD$.

14. The arcs intercepted between a diameter and a parallel chord are equal.

Suggestion. — Draw radii to the ends of the chord, forming central angles which intercept the arcs. The arcs may be proved equal by first proving what? The proof that the central angles are equal depends upon the proof that the triangle formed by the chord and the radii is what kind of triangle?

152. Theorem. — *Through three points not in a straight line one and only one circle can be drawn.*



Hypothesis. A, B, and C are any three points not in a straight line.

Conclusion. Through A, B, and C one and only one circle can be drawn.

Proof. 1. Draw AB, AC, and BC. Draw the perpendicular bisectors of AC and BC, and let them intersect at O.

2. Then O is equidistant from A, B, and C. § 112

3. ∴ a circle with center O and radius OA can be drawn through A, B, and C. § 103

4. The center of any circle passing through A, B, and C is equidistant from A, B, and C. Def. circle

5. ∴ the center must be at O, and hence OA must be the radius. § 105

6. ∴ only one circle can be drawn through A, B, and C. § 151, (3)

153. Corollary 1. — *Two circles can intersect in only two points.*

The proof is left to the student.

154. Corollary 2. — *A straight line can intersect a circle in only two points.*

The proof is left to the student.

155. Corollary 3. — *The perpendicular bisector of a chord passes through the center.*

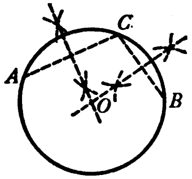
The proof is left to the student.

EXERCISES

1. Construct a circle which shall pass through the three vertices of a given triangle.

2. Having given an arc AB of a circle, complete the circle.

Suggestion. — Draw any two chords, AC and BC, of the arc. Draw the perpendicular bisectors of these chords. How is the center of the circle obtained? The radius?



3. From the measurements of a piece of a broken wheel a new wheel is to be cast of the same size. Show how to find the radius of the new wheel.

4. Draw a circle which shall pass through two given points and have its center on a given line. How is the center found? The radius? When is the construction impossible?

5. Draw a circle of given radius which shall pass through two given points. When is the construction impossible?

6. By marking around some circular object, such as a coin or an ink bottle, draw a circle. Then locate its center.

7. Pirates buried treasure 75 ft. from a certain tree and 100 ft. from a certain straight path. Show how to locate the treasure. When are there two possible locations? When only one? When none?

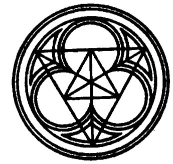
8. If the treasure had been buried equidistant from a certain oak and a certain chestnut and 30 yd. from a certain straight road, show how to locate it.

9. If two perpendicular chords bisect each other, each must be a diameter.

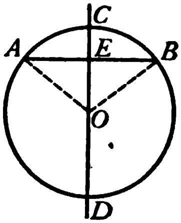
10. If the vertices of a quadrilateral lie on a circle, the perpendicular bisectors of all of the sides intersect at one point.

11. Why cannot a circle be drawn through three points which lie on a straight line?

12. The adjoining design, used in ornamental windows, etc., may be constructed by first drawing a circle through the vertices of a given equilateral triangle. Explain the construction. Make a large drawing similar to this one.



156. Theorem. — *A line through the center of a circle perpendicular to a chord bisects the chord and the subtended arcs.*



Hypothesis. AB is a chord of a circle with center O. Line CD passes through O, intersecting the circle at C and D. And $CD \perp AB$ at E.

Conclusion. $AE = BE$, $\overset{\frown}{AC} = \overset{\frown}{BC}$, and $\overset{\frown}{AD} = \overset{\frown}{BD}$.

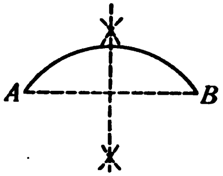
Suggestions. For proving $AE = BE$, what auxiliary lines must be drawn? How are arcs proved equal?

Write the proof in full.

157. Corollary. — *A diameter bisects the circle.*

Suggestion. Show that $\overset{\frown}{AC} + \overset{\frown}{AD} = \overset{\frown}{BC} + \overset{\frown}{BD}$, etc.

158. Construction. — *Bisect a given arc.*



Given $\overset{\frown}{AB}$.

Required to bisect $\overset{\frown}{AB}$.

Suggestion. Draw the chord AB, and draw the perpendicular bisector of this chord. Prove that it bisects $\overset{\frown}{AB}$. Use § 155 and § 156.

Write the construction and proof in full.

EXERCISES

1. The line joining the center of a circle to the middle point of a chord is perpendicular to the chord.

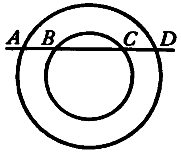
2. A radius which bisects an arc bisects the chord which subtends it and is perpendicular to the chord.

3. A line which bisects a chord and also its subtended arc passes through the center of the circle and is perpendicular to the chord.

4. If a diameter passes through the middle points of two chords of a circle, the two chords are parallel.

5. Two chords which are drawn from the same point on the circle and make equal angles with the diameter through that point are equal.

6. If a straight line intersects two concentric circles in the successive points A, B, C, D, then $AB = CD$.

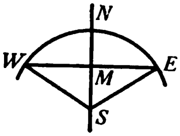


7. Two chords perpendicular to a third chord at its extremities are equal.

SUGGESTION. — Draw a perpendicular from the center of the circle to each of the three given chords. The proof will follow by aid of § 156 if what line-segments are first proved equal?

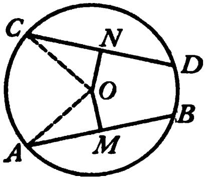
8. Through a given point within a circle draw a chord which shall be bisected at the point.

9. A meridian (a due north and south line) is located by means of equal shadows from the sun, as follows: A plumb line is supported at S. As the day progresses the shadow of the plumb line moves from SW to SE. A circle with center S is drawn on the ground. The points W and E where the shadow crosses the circle are marked, the two observations being made at equal intervals before and after noon (say at 10.30 A.M. and 1.30 P.M.). On June 21 and December 21 the line WE is a due east and west line. Show that if M is the middle point of WE on either of these dates, and SN drawn through M, then SN is the meridian.



10. The locus of the middle points of a set of parallel chords is a diameter.

159. Theorem. — *In the same circle or in equal circles, two equal chords are equidistant from the center.*



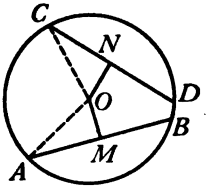
Hypothesis. In circle with center O, chords AB and CD are equal, and $OM \perp AB$ and $ON \perp CD$.

Conclusion. $OM = ON$.

Suggestions. For proving $OM = ON$, what auxiliary lines must be drawn? Show that $AM = CN$ by § 156 and Ax. V. The proof would be the same if the chords were in equal circles.

Write the proof in full.

160. Theorem. — *In the same circle or in equal circles, two chords equidistant from the center are equal.*



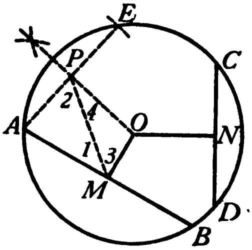
Hypothesis. In circle with center O, AB and CD are chords, $OM \perp AB$, $ON \perp CD$, and $OM = ON$.

Conclusion. $AB = CD$.

Suggestions. For proving $AM = CN$, what auxiliary lines must be drawn? Having proved $AM = CN$, show that $AB = CD$ by § 156 and Ax. IV. The proof applies equally when the chords are in equal circles.

Write the proof in full.

161. Theorem. — *In the same circle or in equal circles, two unequal chords are unequally distant from the center, the greater chord being at the less distance.*



Hypothesis. In circle with center O, AB and CD are chords, $AB > CD$, $OM \perp AB$, and $ON \perp CD$.

Conclusion. $OM < ON$.

Proof. 1. $AB > CD$, $OM \perp AB$, $ON \perp CD$. Hyp.

2. Draw chord $AE = CD$, draw $OP \perp AE$, and draw PM.

3. $AB > AE$. Ax. XII

4. $AM = \frac{1}{2} AB$ and $AP = \frac{1}{2} AE$. § 156

5. $\therefore AM > AP$. Ax. VII

6. $\therefore \angle 1 < \angle 2$. § 143

7. $\angle AMO$ and $\angle APO$ are right angles and hence equal. Def. $\perp$

8. $\therefore \angle 3 > \angle 4$. Ax. VIII

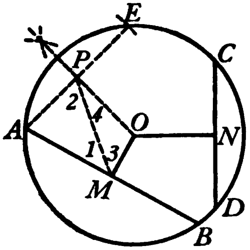
9. $\therefore OM < OP$. § 144

10. But $OP = ON$. § 159

11. $\therefore OM < ON$. Ax. XII

If the chords were in equal circles, the proof would be similar.

162. Theorem. — *In the same circle or in equal circles, two chords unequally distant from the center are unequal, the chord at the less distance being the greater.*



Hypothesis. In circle with center O, AB and CD are chords, $OM \perp AB$, $ON \perp CD$, $OM < ON$.

Conclusion. $AB > CD$.

Suggestions. Draw $AE = CD$, $OP \perp AE$, and draw PM. Proceed with a direct proof.

A short indirect proof may be given also, based upon § 159 and § 161.

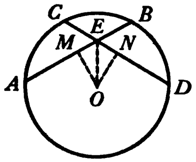
Write out a complete proof.

EXERCISES

1. A diameter is the longest chord of a circle.

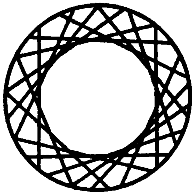
2. The chord drawn through a given point within a circle, perpendicular to the diameter through that point, is the shortest chord which can be drawn through that point.

3. If two chords which intersect within the circle make equal angles with the radius drawn through the point of intersection, the chords are equal.



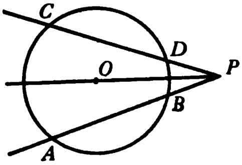
4. If the vertices of a polygon lie on a circle and the sides are equidistant from the center, it is equilateral.

5. If two equal chords intersect, the segments of one are equal to the corresponding segments of the other. That is, if $AB = DC$, prove that $BE = CE$ and $AE = DE$.



6. In this circle, a large number of equal chords are drawn. Explain why they appear to enclose a smaller circle with the same center.

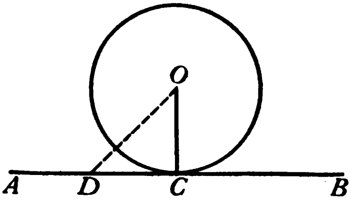
7. Why is a board sawed from a circular log narrower than another board sawed from the same log nearer the center?



8. If PA and PC are two secants which make equal angles with the line connecting P to the center O, chords AB and CD which the circle cuts from the secants are equal.

9. The locus of the middle points of all of the equal chords of a circle is a concentric circle.

163. Theorem. —*A straight line perpendicular to a radius at its outer extremity is tangent to the circle.*



Hypothesis. OC is a radius of circle with center O, and $AB \perp OC$ at C.

Conclusion. AB is tangent to the circle.

Proof. 1. Let D be any point of AB except C. Draw OD.
2. Then $OD > OC$. § 145
3. $\therefore$ D is outside of the circle. § 151, (8)
4. $\therefore$ since C is the only point of AB that is on the circle, AB is tangent to the circle. Def. tangent

164. Theorem. — *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Suggestions. Let AB be tangent at C and let OC be a radius. Take D any point of AB except C. Show that $OD > OC$ by § 151, (8), etc.

Write the proof in full.

EXERCISES

1. The perpendicular to a tangent at the point of contact passes through the center of the circle.

SUGGESTION. — Use indirect proof. Draw the radius to the point of contact. How many perpendiculars can be drawn to a given line at a given point?

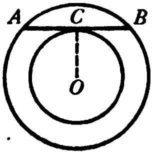
2. A perpendicular from the center of a circle to a tangent passes through the point of contact.

SUGGESTION. — Draw the radius to the point of contact.

3. The tangents to a circle at the extremities of any diameter are parallel.

4. The diameter of a circle bisects all chords parallel to the tangent at the extremity of the diameter.

5. The locus of the centers of circles tangent to a straight line at a given point is a straight line perpendicular to the line at that point.



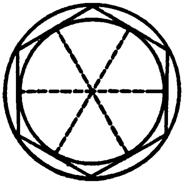
6. If two circles are concentric, a chord of the larger which is tangent to the smaller is bisected at the point of contact.

7. If two circles are concentric, chords of the larger which are tangent to the smaller are equal.

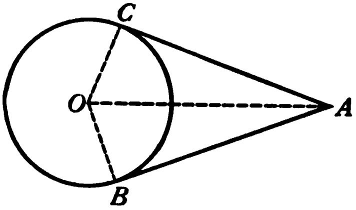
8. All equal chords of a circle are tangent to a concentric circle.

9. If the vertices of an equilateral hexagon are on a circle, a concentric circle can be drawn to which all sides of the hexagon are tangent.

SUGGESTION. — The exercise may be proved if it can be proved that perpendiculars from the center to the sides of the hexagon are equal. By what theorem may the sides then be proved tangents?



165. Theorem. — *The two tangents to a circle from a point outside of the circle are equal.*

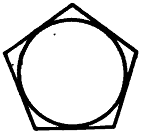


Hypothesis. AB and AC are tangents to circle with center O at B and C, respectively.

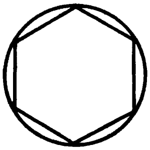
Conclusion. $AB = AC$.

Suggestions. It may be proved that $AB = AC$ if it can be shown that AB and AC are corresponding sides of congruent triangles. It may be proved that $\triangle AOC \cong \triangle AOB$ if it is first proved that $\angle C$ and $\angle B$ are right angles and that $OC = OB$. Hence, begin by drawing OA, OB, and OC, proving that $\angle C$ and $\angle B$ are right angles, etc.

166. Circumscribed and inscribed polygons. — A polygon all of whose sides are tangent to a circle is said to be **circumscribed** about the circle, and the circle is said to be **inscribed** in the polygon.



A CIRCUMSCRIBED POLYGON AND AN INSCRIBED CIRCLE



AN INSCRIBED POLYGON AND A CIRCUMSCRIBED CIRCLE

A polygon all of whose vertices are on a circle is said to be **inscribed** in the circle, and the circle is said to be **circumscribed** about the polygon.

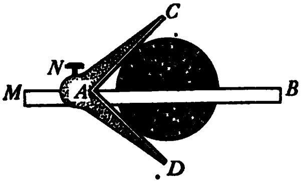
EXERCISES

1. If two tangents are drawn from a point to a circle, the line joining the point to the center of the circle bisects the angle between the tangents.

2. In the figure of § 165, AO bisects $\overset{\frown}{BC}$.

3. In the figure of § 165, AO is the perpendicular bisector of the chord joining the points of contact of the tangents.

4. The instrument called a *center square* is used for locating the centers of circular objects. It consists of a steel blade M, upon which slides an attachment N with two prongs, AC and AD. The edge AB of the blade M bisects the angle between the prongs AC and AD. When the center of a circular object is to be found, the instrument is placed so that the prongs AC and AD are tangent to it. Prove that when this is done AB passes over the center of the object.

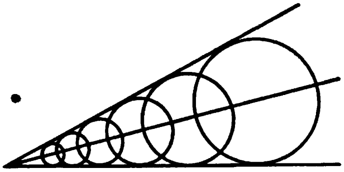


5. The prongs AC and AD of the center square in Ex. 4 are equal. Prove that if the circular object is so large that when the instrument is applied to it the ends of the prongs, C and D, rest against the object, AC and AD becoming secants, then AB passes over the center of the object.

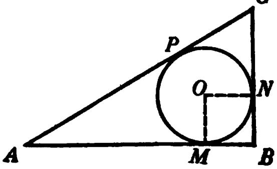
6. The bisectors of the angles of a circumscribed polygon pass through the center of the circle.

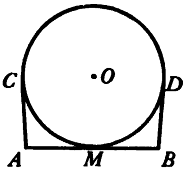
7. The locus of the centers of all circles tangent to both sides of an angle is the bisector of the angle.

SUGGESTION. — Endeavor to apply the theorem in § 108.



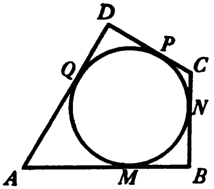
8. If a circle is inscribed in a right triangle, the sum of the legs equals the sum of the hypotenuse and the diameter of the circle.





9. AB is tangent to the circle with center O at M. AC and BD are tangent at C and D, respectively. Prove that $AB = AC + BD$.

10. If an isosceles triangle is circumscribed about a circle, the base is bisected at the point of contact.



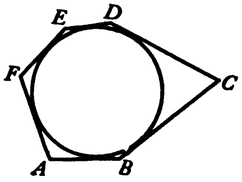
11. If a circle is inscribed in an equilateral triangle, the three sides are bisected at the points of contact.

12. In any circumscribed quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.

13. The perimeter of any circumscribed trapezoid is equal to four times the line-segment which joins the middle points of the two non-parallel sides.

Suggestion. — See § 98.

14. A parallelogram circumscribed about a circle is either a rhombus or a square.



15. In any circumscribed hexagon, the sum of one set of alternate sides equals the sum of the other set, that is

$$AB + CD + EF = BC + DE + FA.$$

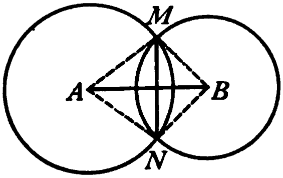
16. In any circumscribed octagon, the sum of one set of alternate sides equals the sum of the other set.

17. The sides of a triangle are 5 in., 6 in., and 8 in. Find the lengths of the segments into which the sides are divided at the points of contact with the inscribed circle.

Suggestion. — Form three equations and solve for the three unknown quantities.

167. Line of centers. — The straight line joining the centers of two circles is called the **line of centers** of the circles.

168. Theorem. — *The line of centers of two intersecting circles is the perpendicular bisector of their common chord.*



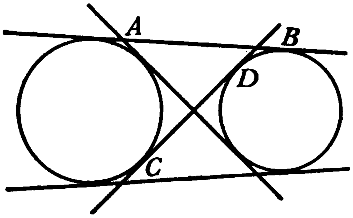
Hypothesis. *A* and *B* are the centers of two intersecting circles, *AB* is the line of centers, and *MN* is the common chord.

Conclusion. *AB* is the perpendicular bisector of *MN*.

Suggestions. The conclusion will follow from § 107, if *A* and *B* are both proved equidistant from *M* and *N*. Hence begin by proving the latter.

Write the proof in full.

169. Common tangents. — A straight line which is tangent to each of two circles is called **a common tangent** of the circles.



If the circles lie on the same side of the common tangent, it is called a common **external** tangent, as *AB*. If the circles lie on the opposite sides of the common tangent, it is called a common **internal** tangent, as *CD*.

EXERCISES

1. What is the largest number of common tangents which two circles may have?

2. Draw a figure showing two circles which have only three common tangents.

3. Draw a figure showing two circles which have only two common tangents.

4. Draw a figure showing two circles which have only one common tangent.

5. Draw two circles which have no common tangent.

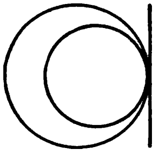
6. Prove that the common internal tangents of two circles are equal.

7. Prove that the common external tangents of two equal circles are equal.

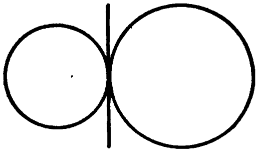
8. Prove that the common external tangents of two unequal circles are equal.

9. What machinery have you seen which illustrates common tangents to two circles?

170. Tangent circles. — Two circles both of which are tangent to the same line at the same point are called **tangent circles**. The point of contact of the circles and line is called the **point of contact** of the circles.



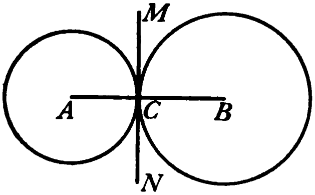
TANGENT INTERNALLY



TANGENT EXTERNALLY

Two circles are said to be tangent **internally** when they lie on the same side of the common tangent line, and tangent **externally** when they lie on opposite sides of it.

171. Theorem. — *If two circles are tangent to each other, the point of contact lies on the line of centers.*



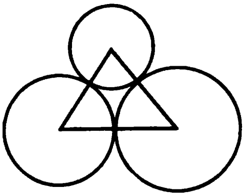
Hypothesis. Circles with centers A and B, respectively, are tangent at C.

Conclusion. C lies on AB.

Suggestions. Let MN be the common tangent at C. Draw AC and BC. It can be proved that C is on AB if it is proved that ACB is a straight line, which in turn requires the proof that $\angle ACB$ is a straight angle.

EXERCISES

1. If two circles are tangent to each other, the distance between their centers equals the sum or difference of their radii.

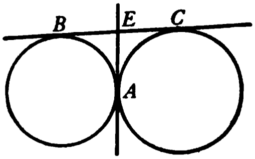


2. Three circles are drawn each tangent to the other two. The lines of centers are 12 in., 16 in., and 18 in., respectively. Find the radii.

SUGGESTION. — Form equations.

3. If two circles are externally tangent at A, the tangents to them from any point in the common tangent at A are equal.

Show that the proof holds also when the circles are internally tangent.



4. Two circles are tangent externally at A, and also have a common tangent touching them at B and C, respectively. Prove that the common tangent at A bisects BC.

5. If two circles are tangent externally at A and have a common tangent touching them at B and C, respectively, a circle with diameter BC will pass through A.

6. In the figure of Ex. 4, $\angle BAC$ is a right angle.

7. If two equal circles are tangent externally at A, and a line is drawn through A intersecting the circles again at B and C, respectively, the chords AB and AC are equal.

SUGGESTION. — Draw the line of centers.

8. A common internal tangent of two equal circles bisects the line of centers.

9. If two equal circles intersect, the common chord bisects the line of centers.

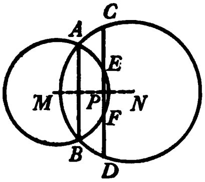
10. If two circles meet on their line of centers, they are tangent.

11. If two circles are externally tangent, and a line is drawn through the point of contact, terminating in the circles, the diameters drawn through the extremities are parallel.

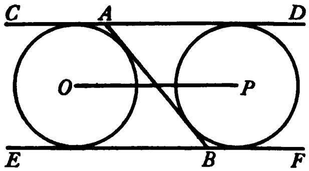
SUGGESTION. — Draw the line of centers.

Show that the proof holds if the circles are internally tangent.

12. If two circles intersect and a straight line is drawn parallel to their common chord and cutting both circles, the segments of the line intercepted between the circles are equal; that is, in the figure, $EC = FD$.



13. If each of two circles is tangent to two parallel lines and also to a transversal of them, the segment of the transversal intercepted between the parallels is equal to the distance between the centers.



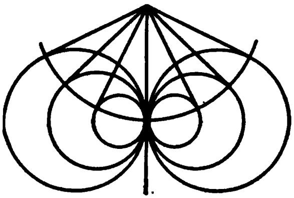
14. If a triangle ABC is formed by the intersection of three tangents to a circle, two of which, AD and AE, are fixed, while the third, BC, touches the circle at a variable point F on the arc DE, prove that the perimeter of the triangle ABC is constant and equal to $AD + AE$.

15. Three equal circles are tangent to each other. Prove that the three common internal tangents meet at a point which is equidistant from the three points of contact.

Suggestion. — Let two tangents meet, and join their point of intersection to the third point of contact. Also join this point of intersection to the centers, and draw the radii to the points of contact.

16. Show that the locus of the centers of all circles tangent to a given circle at a given point is the straight line determined by the given point and the center of the given circle.

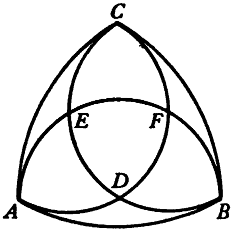
17. Circles are drawn tangent to a given line at the same point of it. From another point of the line tangents are drawn to all of the circles. Prove that the locus of the points of contact of these tangents is a circle.

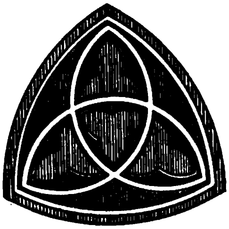


18. What is the locus of the centers of all circles with a given radius r and tangent externally or internally to a given circle? Prove the answer.

19. The adjoining figure is used much in different decorative designs, such as ornamental church windows. Arcs AB, AC, and BC are drawn with centers at C, B, and A, the vertices of an equilateral triangle. Arcs $ADFC$, $BDEC$, and $AEFB$ are semicircles.

Where are the centers of these semicircles?



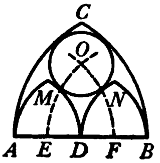


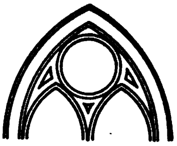
Prove that each semicircle is tangent to two of the arcs, AB, AC, and BC.

Make a large drawing of a window similar to this one.

20. Two equal circles with centers M and N are tangent at A. The lines PM and PN, joining any point P in the common tangent at A to the centers, cut the circles at B and C, respectively. Prove that $PB = PC$.

21. This figure shows a Gothic window. It contains two arcs, $\overset{\frown}{BC}$ and $\overset{\frown}{AC}$, drawn with centers at A and B, respectively, vertices of equilateral $\triangle ABC$. It contains four smaller arcs, $\overset{\frown}{AM}$, $\overset{\frown}{DM}$, $\overset{\frown}{DN}$, and $\overset{\frown}{BN}$. The circle with center O is drawn with radius equal to AE.



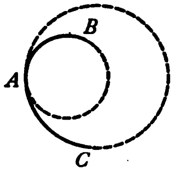


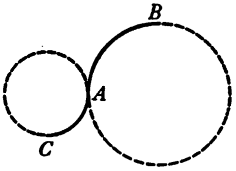
What are the centers and radii of $\overset{\frown}{AM}$, $\overset{\frown}{DM}$, $\overset{\frown}{DN}$, and $\overset{\frown}{BN}$?

How is the center O located?

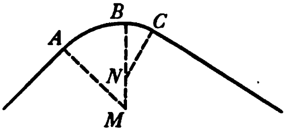
Prove the circle with center O tangent to $\overset{\frown}{AC}$, $\overset{\frown}{BC}$, $\overset{\frown}{DM}$, and $\overset{\frown}{DN}$.

22. Arcs AB and AC of two circles, respectively, which are tangent to each other at A form a *compound curve*. The point of contact A is the transition point from one circle to the other.





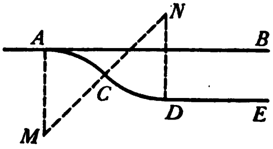
Compound curves are used in the construction of railroads where the winding track must be made to conform to the physical features of the land. They also secure "easement," preventing a train from lurching when it comes to a curve. The curve ABC of a railroad track is composed of two arcs, $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$. The center N of $\overset{\frown}{BC}$ is on the radius MB of $\overset{\frown}{AB}$. Prove that ABC is a *compound* curve, or that $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are arcs of tangent circles.



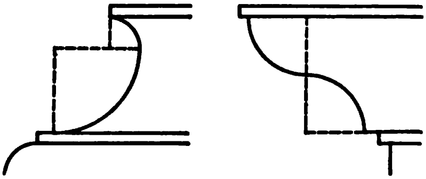
SUGGESTION.—Draw a line through B perpendicular to BM.

23. AB is a straight railroad track. The track of a switch $ACDE$ is laid out as follows: Arc AC is constructed with center M. Then MC is produced to N, making $CN = MC$, and $\overset{\frown}{CD}$ drawn with center N.

Prove that ACD is a *compound* curve, or that $\overset{\frown}{AC}$ and $\overset{\frown}{CD}$ are arcs of tangent circles.



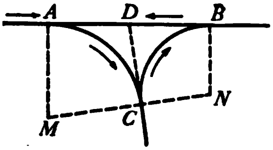
24. Compound curves are used in many kinds of molding, as well as in other architectural designs.



Explain the construction of the curves in the adjoining drawings of moldings.

Make a large drawing of such moldings. Draw designs of other moldings in which compound curves are used.

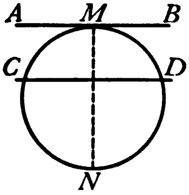
25. A railroad Y consists of three tracks, AC, CB, and AB, upon which a train is reversed in direction, by moving as shown by the arrows, backing from C to B. In constructing a Y, AB is a straight track, and point A given. It is required to locate the curves AC and CB with given radii R and r, respectively.

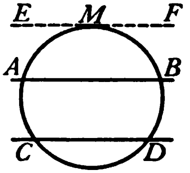


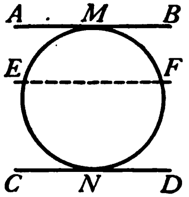
Prove that $AD = DC = DB = \sqrt{R \times r}$.

Suggestion. — Draw MD and ND. Prove $\triangle MCD \sim \triangle NCD$.

172. Theorem. — *Two parallel lines intercept equal arcs on a circle.*







Case I. *When the parallels are a tangent and a secant.*

Hypothesis. AB is tangent to a circle at M; CD is a secant parallel to AB.

Conclusion. $\overset{\frown}{CM} = \overset{\frown}{DM}$.

Proof. 1. AB is tangent at M; secant $CD \parallel AB$. Hyp.

2. Draw diameter MN.

3. Then $MN \perp AB$. § 164
4. $\therefore MN \perp CD$. § 34
5. $\therefore \overset{\frown}{CM} = \overset{\frown}{DM}$. § 156

Case II. *When both of the parallels are secants.*

Hypothesis. Secants AB and CD are parallel.

Conclusion. $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Suggestion. Draw EF parallel to AB and tangent at a point M. Apply Case I.

Case III. *When both of the parallels are tangents.*

Hypothesis. AB is tangent at M and CD tangent at N; $AB \parallel CD$.

Conclusion. Arc MEN = arc MFN.

Suggestion. Draw secant EF parallel to AB. Apply Case I.

Write the proof of the theorem in full.

EXERCISES

1. Prove that the two non-intersecting chords which join the extremities of two parallel chords are equal.

2. Prove that the two intersecting chords which join the extremities of two parallel chords are equal.

3. Prove that an inscribed trapezoid is isosceles.

4. Prove that the diagonals of an inscribed trapezoid are equal.

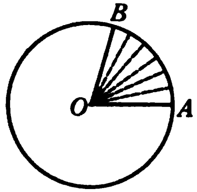
5. Prove, by using Case III of the theorem in § 172, that a diameter of a circle bisects the circle.

6. Prove that if a tangent and a secant intercept equal arcs of a circle, they are parallel.

7. Prove that if two secants which do not intersect within the circle intercept equal arcs, they are parallel.

8. Prove that the opposite sides of an inscribed equilateral hexagon are parallel.

173. The relations of angles and arcs. — If a central angle AOB is divided into unit angles, such as degrees, by drawing radii, then $\overset{\frown}{AB}$ is divided into an equal number of equal arcs. Why? If one of these equal arcs is taken as the unit of measure of arcs, then the *numerical measure* of $\overset{\frown}{AB}$ is the same as that of $\angle AOB$.



This relation may be proved to be equally true when the units of measure are not contained exactly in $\angle AOB$ and $\overset{\frown}{AB}$, that is, when the numerical measures of $\angle AOB$ and $\overset{\frown}{AB}$ can be expressed only approximately. The proof is omitted from this course.

In general, if the unit of measure of arcs is chosen as the arc intercepted by a unit central angle,

A central angle has the same numerical measure as its intercepted arc.

174. Corollary. — *In the same circle or equal circles, any two central angles have the same ratio as their intercepted arcs.*

This follows from the principle in § 173 by definition of ratio. See § 116.

175. Degrees, minutes, and seconds of arc. — An arc which a central angle of one degree intercepts is called a **degree of arc**. Similarly, the arcs which central angles of one minute and of one second intercept are called a **minute of arc** and a **second of arc**, respectively.

Hence, the number of *degrees*, *minutes*, and *seconds* of arc in an arc equals the number of degrees, minutes, and seconds in the central angle which intercepts it.

Thus, a central angle of $62° 15' 30''$ intercepts an arc of $62° 15' 30''$.

It follows that an *entire circle* equals 360°, a *semicircle* equals 180°, and a *quadrant* equals 90° of arc.

In all work that follows, arcs will be measured in degrees, minutes, and seconds.

Two arcs whose sum is 90° are called **complementary** arcs, and two arcs whose sum is 180° are called **supplementary** arcs.

EXERCISES

1. Find the complement of an arc of 47° 24′ 40″.

2. Find the supplement of an arc of 125° 52′ 15″.

3. If a circle is divided into three arcs, two of which are 120° 30′ 15″ and 110° 40′ 30″, respectively, find the number of degrees, minutes, and seconds in the third arc.

4. Prove that in the same circle or equal circles the complements of equal arcs are equal.

5. Prove that in the same circle or equal circles the supplements of equal arcs are equal.

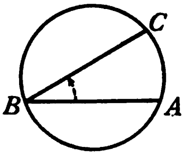
6. How many degrees are there in the arc subtended by each side of an inscribed equilateral hexagon? Octagon?

7. If equilateral $\triangle ABC$ is inscribed in a circle with center O, how many degrees are there in $\widehat{AB}$? In $\angle AOB$?

8. If AB is a side of an equilateral pentagon inscribed in a circle with center O, how many degrees are there in $\widehat{AB}$? In $\angle AOB$?

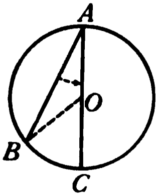
9. Has a degree of arc the same length in two unequal circles?

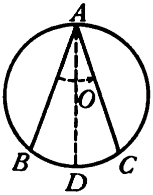
176. Inscribed angles. — An angle is said to be **inscribed in a circle** when its vertex is on the circle and its sides are chords, as $\angle ABC$. Angle ABC *intercepts* $\widehat{AC}$.

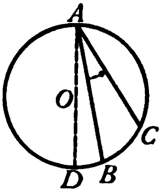


An angle is said to be **inscribed in an arc** when its vertex is on the arc and its sides pass through the extremities of the arc. Thus, $\angle ABC$ is inscribed in arc ABC.

177. Theorem. — *An inscribed angle of a circle has the same measure as one half of its intercepted arc.*







CASE I. *When the center is on one side of the angle.*

Hypothesis. $\angle BAC$ is inscribed in circle with center O, and O is on AC.

Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}\ \overset{\frown}{BC}$.

Proof. 1. $\angle BAC$ is inscribed and center O is on AC. Hyp.

2. Draw OB.

3. Then $\angle BAC + \angle OBA = \angle BOC$. § 51

4. But $OB = OA$. § 151, (2)

5. $\therefore \angle OBA = \angle BAC$. § 70

6. $\therefore 2\angle BAC = \angle BOC$. Ax. XII

7. $\therefore \angle BAC = \frac{1}{2}\angle BOC$. Ax. V

8. But $\angle BOC$ has the same measure as $\overset{\frown}{BC}$. § 173

9. $\therefore \angle BAC$ has the same measure as $\frac{1}{2}\ \overset{\frown}{BC}$. Ax. V

CASE II. *When the center is within the angle.*

Hypothesis. $\angle BAC$ is inscribed and center O is within $\angle BAC$.

Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}\ \overset{\frown}{BC}$.

Suggestions. Draw diameter AD. Apply Case I to $\angle BAD$ and $\angle DAC$, and add.

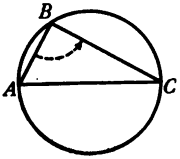
CASE III. *When the center is outside of the angle.*

Hypothesis. $\angle BAC$ is inscribed and center O is outside of $\angle BAC$.

Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}\overset{\frown}{BC}$.

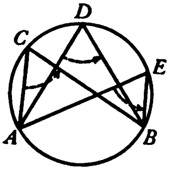
Suggestions. Draw diameter AD. Apply Case I to $\angle DAB$ and $\angle DAC$, and subtract.

Write the proof of the theorem in full.



178. Corollary 1. — *An angle inscribed in a semicircle is a right angle.*

The proof is left to the student.



179. Corollary 2. — *All angles inscribed in the same arc are equal.*

The proof is left to the student.

EXERCISES

1. In the figure of Case I, § 177, if $\overset{\frown}{BC}$ is one fourth of the entire circle, how many degrees in $\angle BAC$?

2. If an angle inscribed in an arc of a circle is a right angle, the arc is a semicircle.

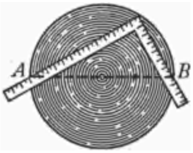
3. An angle inscribed in an arc less than a semicircle is obtuse.

4. An angle inscribed in an arc greater than a semicircle is acute.

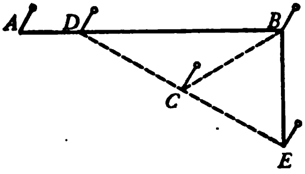
5. If an angle inscribed in an arc is obtuse, the arc is less than a semicircle.

6. If an angle inscribed in an arc is acute, the arc is greater than a semicircle.

7. Prove that the center of any circular object may be located by means of a carpenter's square as follows: Lay the square on the object, with the heel at the rim, and mark the points A and B where the blades cross the rim. Now, by placing the blade of the square on A and B, find the middle point of AB. That is the center.



8. Prove that the following method may be used by a surveyor to lay out a line perpendicular to *AB* at *B*: Select any point *C* 50 ft. from *B*. With one end of a 50-ft. tape at *C*, swing the other end to *D*, in line with *A* and *B*. With one end of the tape still at *C*, swing the other end to locate *E* in line with *D* and *C*. Then *BE* is the required perpendicular.

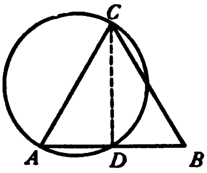


Suggestion. — A circle may be drawn through *D*, *B*, and *E*.

9. The bisector of an inscribed angle bisects the intercepted arc.

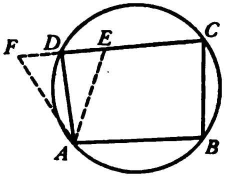
10. The bisectors of all angles inscribed in a given arc of a circle are concurrent.

11. If a circle is drawn with one of the equal sides of an isosceles triangle as its diameter, it bisects the base.



12. The opposite angles of an inscribed quadrilateral are supplementary.

13. If the opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about it.



Suggestion. — Draw a circle passing through *A*, *B*, and *C*, and prove by indirect method that it passes through *D*. Show that it cannot cut *DC* at *E*, between *D* and *C*, nor *DC* produced at *F*. See § 51.

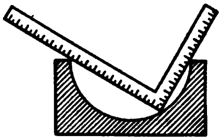
14. A parallelogram inscribed in a circle is a rectangle.

15. If *ABCD* is an inscribed square, and *M* any point on $\widehat{AB}$, then *MC* and *MD* trisect (divide into three equal parts) $\angle AMB$.

16. Chords *AB* and *CD* intersect within the circle at *M*. Prove that $\triangle BCM$ and $\triangle ADM$ are mutually equiangular.

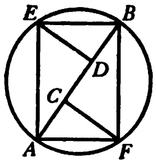
17. Chords *AB* and *CD* extended meet outside of the circle at *M*. Prove $\triangle BCM$ and $\triangle ADM$ mutually equiangular.

18. Pattern makers and others use the carpenter's square as follows to determine if a half-round hole is a true semicircle: The square is placed as in the figure. If the heel of the square touches the bottom of the hole in all positions

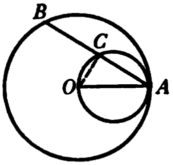


of the square, while the blades rest against the edges of the hole, the hole is a true semicircle. Prove that the method is accurate.

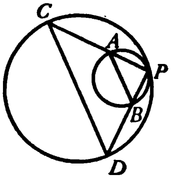
19. The strongest beam that can be cut from a round log has a cross section which may be constructed as follows: Draw diameter AB and trisect it at C and D. Draw $ED \perp AB$ and $CF \perp AB$. Draw AE, EB, BF, and FA. Prove that $AEBF$ is a rectangle.



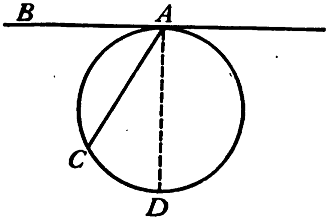
20. A circle is drawn with the radius of another circle as its diameter. Through A, their point of contact, any chord AB of the larger circle is drawn, intersecting the smaller circle at C. Prove that C bisects AB.



21. Two circles are internally tangent at P. AB is a diameter of the smaller circle. PA and PB intersect the outer circle at C and D, respectively. Prove that CD is a diameter.



180. Theorem. — *An angle formed by a tangent and a chord drawn from the point of contact has the same measure as one half of the intercepted arc.*

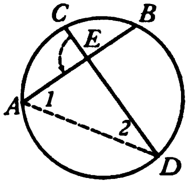


Hypothesis. AB is a tangent at A, and AC is a chord.

Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}\overset{\frown}{AC}$.

Suggestions. Draw diameter AD. Since $\angle BAD = 90°$ and arc $ACD = 180°$, $\angle BAD$ has the same measure as $\frac{1}{2}$ arc ACD. What measure has $\angle CAD$? Write the proof.

181. Theorem. — *An angle formed by two chords intersecting within the circle has the same measure as one half of the sum of the arcs intercepted by its sides and the sides of the vertical angle.*



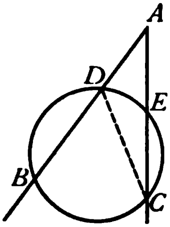
Hypothesis. AB and CD are chords intersecting at E.

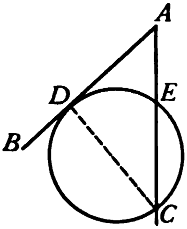
Conclusion. $\angle CEA$ has the same measure as $\frac{1}{2}(\overset{\frown}{AC}+\overset{\frown}{BD})$.

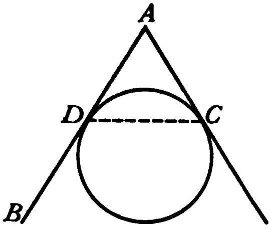
Suggestions. Draw AD. $\angle CEA = \angle 1 + \angle 2$. Why? $\angle 1$ and $\angle 2$ have the same measures as what?

Write the proof in full.

182. Theorem. — *An angle formed by two secants, or a secant and tangent, or two tangents, intersecting outside of the circle has the same measure as one half of the difference of its intercepted arcs.*







Suggestion. In each figure, draw DC, and compare $\angle A$ with $\angle BDC$ and $\angle ACD$.

Write the proof in full.

EXERCISES

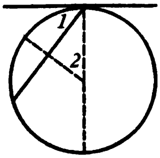
1. Tangents drawn at the extremities of a chord of a circle make equal angles with the chord.

2. If a chord is drawn from the point of contact of a tangent to a circle, the line joining the middle point of the intercepted arc to the point of contact bisects the angle between the tangent and chord.

3. If two chords are perpendicular to each other, the sum of one pair of arcs intercepted by vertical angles equals the sum of the other pair, and each equals a semicircle.

4. AB is a diameter of a circle with center O, and AC is a chord. If $OD \parallel AC$, prove that OD bisects $\widehat{BC}$.

5. The bisectors of the angles of an equilateral triangle pass through the center of the circumscribed circle.



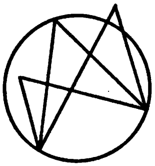
6. Prove the theorem in § 180 by drawing a radius perpendicular to the chord and the diameter from the point of contact of the tangent. ($\angle 1 = \angle 2$. Why?)

7. Prove the theorem in § 180 by drawing the chord from C parallel to AB.

8. Prove the theorem in § 181 by drawing a line through an extremity of one of the chords and parallel to the other.

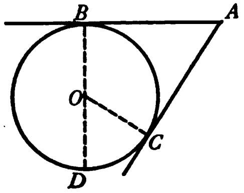
9. Prove the theorem in § 182 by drawing a line through D parallel to AC.

10. An inscribed angle of a circle is less than an angle whose vertex is within the circle and whose sides intercept the same arc, and greater than an angle whose vertex is outside of the circle and whose sides intercept the same arc.



11. If an angle is greater than the inscribed angle of a circle which intercepts the same arc, its vertex is within the circle; if it is equal to the inscribed angle, its vertex is on the circle; and if it is less than the inscribed angle, its vertex is outside of the circle.

Suggestion. — Use the indirect method in proving each of these parts.

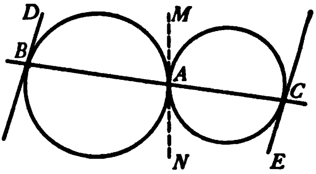


12. The angle formed by two tangents has the same measure as the supplement of the smaller of the two intercepted arcs.

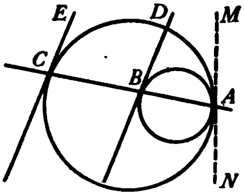
Suggestion. — Prove $\angle A = \angle DOC$.

13. If two circles are externally tangent and a line is drawn through the point of contact intersecting the circles, tangents to the circles at the points of intersection are parallel.

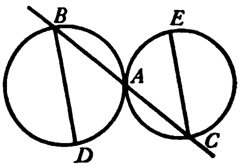
SUGGESTION. — Draw the common internal tangent. What is the relation of $\angle ABD$ and $\angle ACE$ to angles at A?



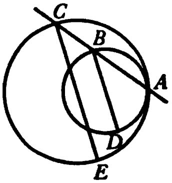
14. If two circles are internally tangent and a line is drawn through the point of contact intersecting the circles, tangents to the circles at the points of intersection are parallel.



15. If two circles are externally tangent and a line is drawn through the point of contact intersecting the circles, diameters drawn from the points of intersection are parallel.



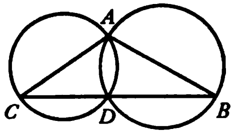
16. If two circles are internally tangent and a line is drawn through the point of contact intersecting the circles, diameters drawn from the points of intersection are parallel.



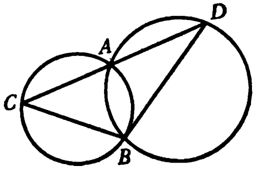
17. If two parallel tangents intercept a segment on a third tangent, the intercepted segment subtends a right angle at the center of the circle.

18. The circles drawn with two sides of any triangle as diameters intersect on the third side.

SUGGESTION. — In the figure, let D be the intersection of the circles. Draw AD, BD, and CD. Prove that CDB is a straight line.



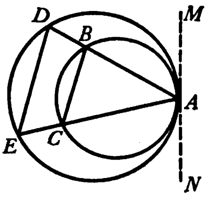
19. Two circles intersect at A and B, and CD is any line-segment drawn through A terminating in the circles. Prove that $\angle DBC$ is constant (has the same size) for all positions of CD.



20. Through A, a point of intersection of two equal circles, two line-segments BC and DE are drawn, terminating in the circles. Prove the chords BD and EC equal.

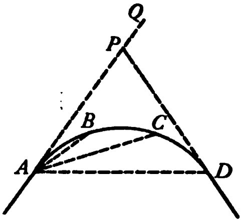
21. In the figure of Ex. 20, if BF and CF, and also DF and EF, are drawn, prove that $BF = CF$ and $DF = EF$.

22. Two circles are tangent internally at A. Chords ABD and ACE are drawn cutting the circles at B, D, C, and E, as in the figure. Prove $DE \parallel BC$.

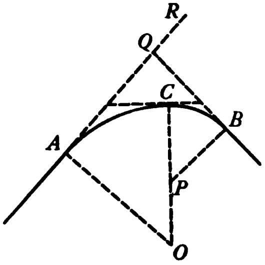


23. Is the principle in Ex. 22 true when the circles are externally tangent? Prove it.

24. In railroad surveying, curves are laid out by turning off equal angles and setting stakes every 100 ft. If the curve begins at A, $\angle PAB$ is turned off from the tangent AP and AB measured 100 ft., then $\angle BAC$ is turned off and BC measured 100 ft., etc., the process being continued until the curve ends in the tangent DP at D. Since the curve is to be an arc of a circle, prove that $\angle PAB$, $\angle BAC$, etc., must be made equal, and that each must equal one half of the central angle subtended by a 100-ft. chord.

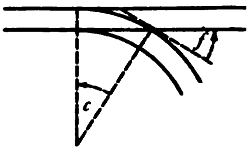


25. In Ex. 24, $\angle QPD$ is called the "intersection" angle, and $\angle PAB$, $\angle BAC$, etc., the "deflection" angles of the curve. Prove that the intersection angle equals twice the sum of the deflection angles. Show that the intersection angle equals the central angle subtended by the curve.

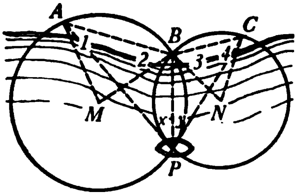


26. The railroad curve ACB is a compound curve, composed of $\widehat{AC}$ and $\widehat{CB}$, with centers at O and P, respectively. Point P is on OC. Prove that the intersection angle BQR equals the sum of the central angles, $\angle AOC$ and $\angle CPB$.

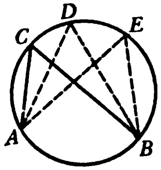
SUGGESTION. — Draw the common tangent at C.



27. In laying out a switch on a railroad track, a "frog" is used at the intersection of the two rails, to allow the flanges of wheels rolling on one rail to cross the other. If one track is straight and the other one curved, prove that the angle f of the frog equals the central angle c of the curve. (Note that $\angle f$ is formed by the straight rail and the tangent to the curved rail at the point where the two rails cross.)

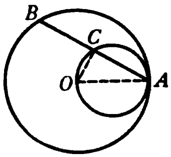


28. An important problem in marine surveying is to determine the position P of a boat from which soundings are being made along a coast. The boat moves from place to place, and it is necessary to locate these positions on the chart. Three stations, A, B, and C, are located on the shore. Angles x and y are observed from the boat. A, B, and C are located on the chart. P is located on the chart by the intersection of two circles, one passing through A and B, and the other through B and C. Show how to locate their centers. Suppose that $\angle x = 40°$, and $\angle y = 70°$.



29. The locus of the vertices of all angles which are equal to a given angle and whose sides pass through two given points is the arc of a circle terminating in the two points and such that an angle inscribed in the arc equals the given angle.

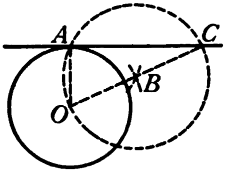
SUGGESTION.—Let $\angle ACB$ be a given angle and A and B two given points. Let D be any point on arc ACB, and prove that $\angle ADB = \angle ACB$. Let $\angle AEB$ be any angle equal to $\angle ACB$ whose sides pass through A and B, and prove indirectly by § 182 that E is on arc ACB.

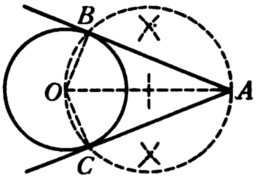


30. The locus of the middle points of all chords which are drawn from the same point on a circle is a circle.

SUGGESTION.—Let C be the middle point of AB, any chord which is drawn from the given point A of the given circle, and O the center of the circle. What kind of angle is $\angle OCA$? Use Ex. 29.

183. Construction. — *Through a given point, draw a tangent to a given circle.*





Case I. *When the given point is on the given circle.*

Given point A on circle with center O.

Required to draw a tangent to the circle at A.

Construction. 1. With centers at the points A and O, and with equal radii of any convenient length, draw arcs intersecting at a point B.

2. With center B and radius OB, draw a circle.

3. Draw the diameter OC of this circle.

4. Draw a straight line through A and C.

5. Then AC is the required tangent.

Proof. The proof is left to the student. It is based on § 163 and § 178.

Case II. *When the given point is outside of the given circle.*

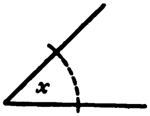
Given point A outside of circle with center O.

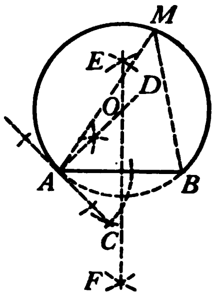
Required to draw a tangent to the circle through A.

Suggestions. How is the circle with diameter OA drawn? How are AB and AC drawn? Why are $OB \perp AB$ and $OC \perp AC$? Use § 163. How many tangents may be obtained?

Make the constructions, and write in full the steps of the construction and proof for each case.

184. Construction. — *Upon a given line-segment as a chord, construct an arc in which an inscribed angle shall equal a given angle.*





Given $\angle x$ and line-segment AB.

Required to construct on AB as a chord an arc in which an inscribed angle shall equal $\angle x$.

Construction. 1. Construct $\angle CAB = \angle x$.

2. Draw $AD \perp AC$.

3. Draw EF, perpendicular bisector of AB, intersecting AD at O.

4. With center O and radius AO, draw a circle.

5. Then arc AMB of this circle is the required arc.

Proof. The proof is left to the student.

Make the construction, and write the steps of the construction and proof in full.

185. Instruments used in geometric construction. — Since the time of the ancient Greeks, the constructions of the *science* of elementary geometry have, by common consent, been limited to those which may be made by the use of the *unmarked straightedge* and *compasses* alone, such as those in § 183, § 184, and earlier sections of the book.

But, as has been seen, in the *practical applications* of geometry, other instruments for measurement, such as the graduated ruler, protractor, draftsman's triangle, etc., are used also.

If, in geometric constructions, the instruments used are limited to the ungraduated straightedge and compasses alone, there are some constructions which cannot be made. There are three such impossible constructions which have interested mathematicians since the time of the ancient Greeks; namely,

(1) *To trisect (cut into three equal parts) any given angle;*

(2) *To construct a square which shall have the same area as a given circle;*

(3) (In solid geometry) *To construct a cube which shall have twice the volume of a given cube.*

All of these constructions are possible by means of other instruments.

186 Method of attack in constructions. — As the problems of construction that have preceded show, every construction depends upon one or more known truths, such as theorems.

In attacking a problem of construction, there is a definite procedure by means of which the construction may be discovered. If the construction is not suggested at once, proceed as follows:

(1) *Suppose the construction made, and make a free-hand drawing of the required figure.*

(2) *In the free-hand drawing, study the relations of the given parts to the remaining parts, and strive to recall theorems, corollaries, etc., involving them.*

(3) *If relations thus discovered suggest the construction, discard the free-hand drawing, and make the construction accurately by means of straightedge and compasses.*

(4) *By means of the theorems, etc., which suggested the con-*

struction, write out a proof that the figure obtained meets all of the requirements.

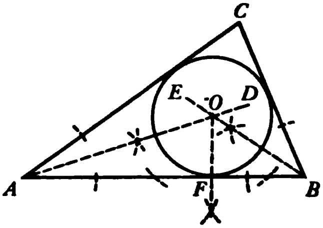
(5) *At the end of the proof discuss the conditions under which more than one figure meeting the requirements may be obtained, or under which no figure meeting the requirements may be obtained.*

The process suggested in steps (1) and (2) above is called **geometric analysis.**

EXERCISES

1. Inscribe a circle in a given triangle.

ANALYSIS. — Since O, the required center, must be equidistant from the sides, it must be on the bisectors of $\angle A$ and $\angle B$, and hence at their intersection. Since AB must be tangent to the circle, it must be perpendicular to the radius OF. Hence:



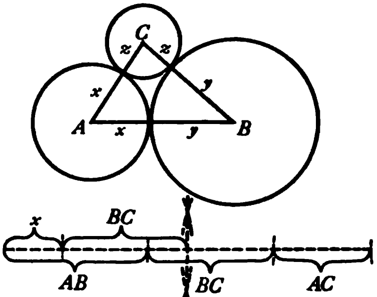
CONSTRUCTION. — 1. Draw AD and BE bisecting $\angle A$ and $\angle B$, respectively, and let them meet at O.

2. Draw $OF \perp AB$.

3. With center O and radius OF, draw a circle.

4. This is the required circle.

Let the student make the construction and write the construction and proof in full. Do not include the analysis in the written work.



2. Construct three circles each tangent to the other two, with the vertices of a given triangle as centers.

ANALYSIS. — If x, y, and z are the required radii,

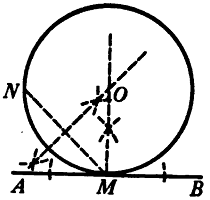
$x + y + z =$ half perimeter.

Also $\quad y + z = BC$.

Hence, $\quad x =$ half perimeter less BC.

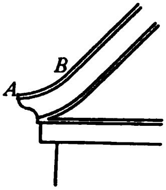
Let the student make the construction and write the complete construction and proof.

3. Construct a circle which shall be tangent to a given line at a given point and pass through a given external point.



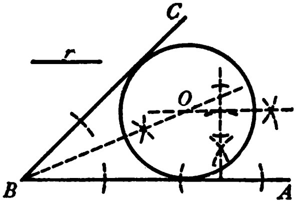
Analysis. — If the required circle is tangent to AB at M, radius $OM \perp AB$. If it passes through M and N, MN is a chord, and hence O is on the perpendicular bisector of MN. Hence O is the intersection of the perpendicular to AB at M and the perpendicular bisector of MN.

Let the student make the construction, and write the complete construction and proof.



4. In architecture it is sometimes required to draw an *easement cornice* tangent to the straight cornice at B, and passing through a given point A. Explain the construction and make such a drawing.

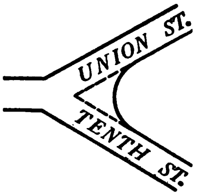
The same construction is used in laying out the easements of stair rails.



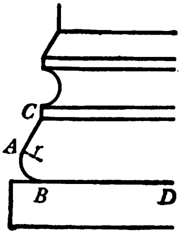
5. Construct a circle with a given radius and tangent to the sides of a given angle.

Analysis. — If r is the given radius and $\angle ABC$ the given angle, the center O must be on a line parallel to AB, at the distance r from AB, and also on the bisector of $\angle ABC$, and hence is their intersection.

Let the student make the construction, and write the construction and proof in full.

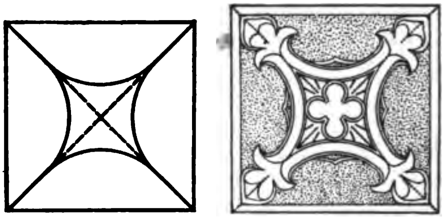


6. Draw the *connecting curve* of given radius r at the intersection of the curbs of two given streets.



7. In this base of a *column*, the lines AC and BD are given, and it is required to connect them by a circular arc with a given radius r. Show how to make the construction, and make a drawing on a large scale of the whole base of a column like this one.

8. The adjoining figure is of a steel *ceiling panel.* The four tangent arcs are drawn with centers on the sides of the square. Construct a similar design three inches square.

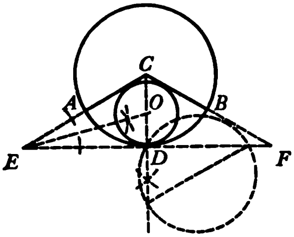


Analysis. — Since the arcs are to be tangent to each other, they must be tangent to what lines? Since each arc is to be tangent to both diagonals, its center must be on what line?

9. Construct a circle tangent to a given arc and to the sides of the central angle which intercepts the arc.

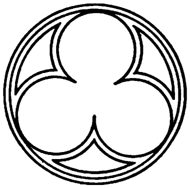
Analysis. — To what point of $\widehat{AB}$ is radius CD drawn? How is EF drawn? How are points E and F located? Then the required circle is inscribed in $\triangle EFC$. See Ex. 1.

Make the construction, and write the complete construction and proof.



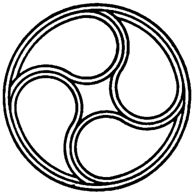
10. In a given circle construct three equal circles, each tangent to the other two and to the given circle.

Suggestion. — Construct three equal angles at the center. Then apply Ex. 9. After the first of the three circles is obtained, how may the other two be constructed most easily?



11. The construction of a *trefoil,* formed by the arcs of three equal circles in a given circle as shown in the figure, is encountered frequently in architectural designs. Explain the construction, and draw a trefoil in a given circle.

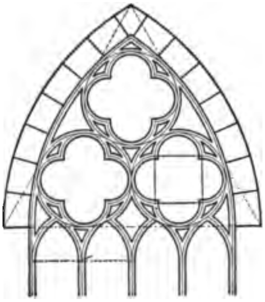
Suggestion. — Use a circle at least three inches in diameter. Proceed as in Ex. 10.



12. Show how to locate the centers of the circular arcs in this *rosette.* Draw a rosette like this three inches in diameter.

Suggestion. — Proceed as in Ex. 10.

13. This *tracery window* contains three equal circles within an equilateral triangle, each circle being tangent to the other two circles and to the two sides of the triangle. Construct three such circles within a given triangle.



SUGGESTION. — Since the circles are tangent to each other, they must have a common tangent line at each point of contact. Then each circle is inscribed in a triangle how found?

14. Construct within a given square four equal circles, each tangent to two others and to two sides of the square.

15. Construct a circle with given radius which shall pass through two given points.

16. Construct a circle with given radius tangent to a given circle and to a given straight line.

17. Construct a circle with given radius which shall be tangent to two given circles.

18. Construct a circle which shall pass through a given point and be tangent to a given circle at a given point.

19. Construct a circle with given radius which shall pass through a given point and be tangent to a given circle.

20. Through a given point draw a straight line cutting a given circle so that the chord intercepted on it by the circle shall equal a given length.

21. In a given circle draw a chord of given length and parallel to a given straight line.

22. Through an intersection of two given circles draw a line-segment of given length terminating in the two circles.

23. Draw a tangent to a given circle which shall be perpendicular to a given straight line.

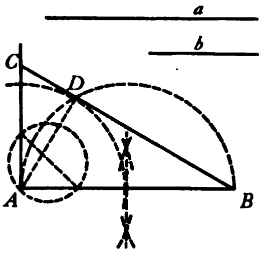
24. Draw a tangent to a given circle which shall be parallel to a given straight line.

Construct a right triangle, having given:

25. One leg and the hypotenuse.

26. One leg and the altitude to the hypotenuse.

ANALYSIS. — If $\triangle ABC$ is the required triangle, with leg AB equal to the given segment a, and AD, altitude to hypotenuse BC, equal to the given segment b, then BC is tangent to a circle with center A and radius AD.



Make the construction, and write out the complete construction and proof.

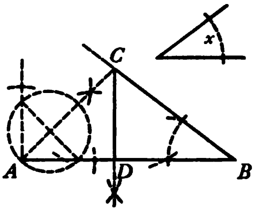
27. The hypotenuse and the altitude to the hypotenuse.

28. An acute angle and the altitude to the hypotenuse.

29. An acute angle and the sum of the legs.

ANALYSIS. — If $\angle x$ is the given angle, AB the sum of the legs, and $\triangle DBC$ the required triangle, then $CD = AD$. Hence $\angle DAC = 45°$. Therefore begin by constructing an angle of 45° at A.

Make the construction, and write the complete construction and proof.



30. The hypotenuse and the sum of the legs.

31. Construct an equilateral triangle, having given an altitude.

32. Construct an isosceles triangle, having given the base and the angle at the vertex.

33. Construct a triangle, having given two sides and the angle opposite one of them.

34. Construct a triangle, having given a side, the angle opposite it, and the altitude to another side.

35. Construct a triangle, having given a side, the median to that side, and the altitude to that side.

36. Construct a triangle, having given two sides and the altitude to one of the given sides.

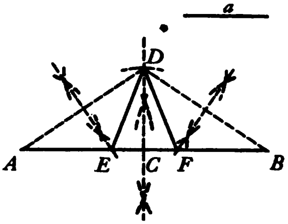
37. Construct a triangle, having given a side and the altitudes to the other two sides.

38. Construct a triangle, having given a side, the altitude to the given side, and the radius of the circumscribed circle.

39. Construct a triangle, having given one side, an adjacent angle, and the sum of the other two sides.

40. Construct a triangle, having given the three medians.

41. Construct an isosceles triangle, having given the perimeter and the altitude to the base.



Aɴᴀʟʏsɪs.—If *a* is the given altitude, *AB* the given perimeter, and *DEF* the required triangle, then *DE* = *AE* and *DF* = *BF*. How, therefore, are *E* and *F* located?

Make the construction and write the construction and proof in full.

42. Draw a tangent to a given circle at a given point when the center is inaccessible.

43. Draw the bisector of a given angle when the vertex is inaccessible.

CHAPTER IX

NUMERICAL RELATIONS

187. Numerical relations. — The *numerical measure* of a geometric figure was defined in § 115. A *proportion* between the parts of similar figures was defined in Chapter VI as a certain *relation between their numerical measures.* In the present chapter certain proportions and certain other numerical relations of geometric figures that are derived from proportions will be considered.

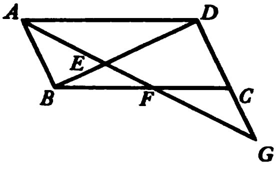
Since the terms of a proportion between the parts of geometric figures are taken as abstract numbers, they may be multiplied or subjected to any other of the operations of abstract numbers.

Thus, in the sections which follow, the *product* of two line-segments AB and CD shall be taken to mean the product of the abstract numbers corresponding to their numerical measures.

EXERCISES

1. In the parallelogram $ABCD$, AG is drawn cutting the diagonal BD at E, side BC at F, and side DC produced at G. Prove that

$$\frac{AE}{EF}=\frac{EG}{AE}.$$

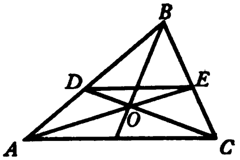


Suggestion. — Prove $\triangle AED \sim \triangle BEF$ and $ABE \sim \triangle EDG$.

2. A median to a side of a triangle bisects any line-segment parallel to that side and terminating in the other two sides.

3. A line-segment parallel to the bases of any trapezoid, terminating in the sides, and drawn through the intersection of the diagonals, is bisected at that point.

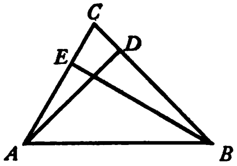
4. In $\triangle ABC$, DE is parallel to the side AC and intersects AB at D and BC at E. If DC and AE intersect at O, prove BO a median of the triangle. Use Ex. 3.



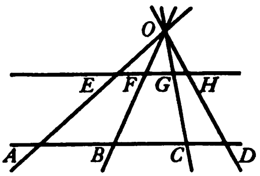
5. If one of the bases of a trapezoid is double the other, the diagonals intersect at a point of trisection of each.

6. State and prove the proportion between two altitudes of any triangle and the sides to which they are drawn.

Suggestion. — If AD and BE are altitudes of $\triangle ABC$, $\frac{AD}{BE} = \frac{AC}{BC}$.



7. If three or more concurrent lines intersect two parallel lines, the corresponding segments of the parallel lines that are intercepted are proportional.



Hypothesis. — AE, BF, CG, etc., are concurrent at O, and $AB \parallel EF$.

Conclusion. — $\frac{AB}{EF} = \frac{BC}{FG} =$ etc.

Suggestion. — The conclusion will follow if it is first proved that $\frac{AB}{EF}$ and $\frac{BC}{FG}$ are each equal to $\frac{OB}{OF}$.

8. If three or more straight lines intersect two parallel lines, making the corresponding intercepted segments of the parallel lines proportional, the lines are either concurrent or parallel.

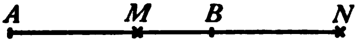
Hypothesis. — In a figure lettered as in Ex. 7, $\frac{AB}{EF} = \frac{BC}{FG} =$ etc.

Conclusion. — AE, BF, CG, etc., are either concurrent or parallel.

Suggestion. — Suppose that no two of these lines are parallel and that AE and BF intersect at O. Draw OC, intersecting EF at M. Prove that M and G coincide, and hence that CG passes through O.

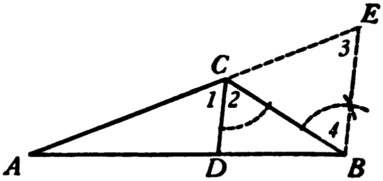
9. The straight line joining the middle points of the bases of any trapezoid and the two non-parallel sides produced are concurrent.

188. Internal and external division of a line-segment. — A line-segment is divided **internally** into two parts at a point between its extremities, and **externally** into two parts at a point on the segment produced through one of its extremities.



Thus AB is divided *internally* at M into AM and MB, and is divided *externally* at N into AN and NB.

189. Theorem. — *The bisector of an angle of a triangle divides the opposite side internally into segments proportional to the adjacent sides.*



Hypothesis. In $\triangle ABC$, CD bisects $\angle ACB$.

Conclusion. $\frac{AD}{DB} = \frac{AC}{BC}$.

Proof. 1. In $\triangle ABC$, CD bisects $\angle ACB$. Hyp.

2. Draw $BE \parallel CD$ and produce AC to meet it at E.

3. Then $\frac{AD}{DB} = \frac{AC}{CE}$. § 122

4. $\angle 1 = \angle 2$. Def. bisector

5. $\angle 3 = \angle 1$. § 26

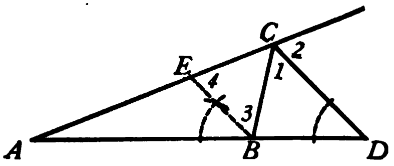
6. $\angle 4 = \angle 2$. § 29

7. $\therefore \angle 3 = \angle 4$. Ax. I

8. $\therefore CE = BC$. § 74

9. $\therefore \frac{AD}{DB} = \frac{AC}{BC}$. Ax. XII

190. Theorem. — *The bisector of an exterior angle of a triangle that intersects the opposite side divides it externally into segments proportional to the adjacent sides.*



Hypothesis. In $\triangle ABC$, CD bisects the exterior angle at C and intersects AB produced at D.

Conclusion. $\dfrac{AD}{DB}=\dfrac{AC}{BC}$.

Suggestions. Draw $BE \parallel CD$, meeting AC at E. Prove that $\dfrac{AD}{DB}=\dfrac{AC}{CE}$. Hence the conclusion may be established if it can be proved that $CE=BC$.

Write the proof in full.

EXERCISES

1. In $\triangle ABC$, $AB=5$ in., $AC=6$ in., and $BC=7$ in. Find the segments into which the bisector of $\angle C$ divides AB.

SUGGESTION. — Let the segment adjacent to AC be x in. Then

$$\frac{x}{5-x}=\frac{6}{7}.$$

2. In the triangle of Ex. 1, find the segments into which the sides AC and BC are divided by the bisectors of the opposite angles.

3. The sides of a triangle are 8 in., 10 in., and 12 in. Find the segments into which the 12-inch side is divided by the bisector of the exterior angle at the opposite vertex.

4. Prove by § 189 that the bisector of the vertical angle of an isosceles triangle bisects the base.

5. When a line-segment is divided internally and externally in the same ratio, it is said to be divided *harmonically*. Prove that the bisectors of the interior and exterior angles at any vertex of a triangle divide the opposite side harmonically.

6. Prove that a straight line which is drawn through a vertex of a triangle and divides the opposite side internally into segments proportional to the adjacent sides bisects the angle at that vertex.

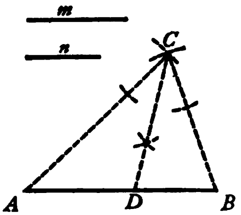
Suggestion. — Using a figure lettered like that of § 189, prove $\angle 1 = \angle 2$. For this purpose, produce AC to E, making $CE = BC$.

A short indirect proof may be given also.

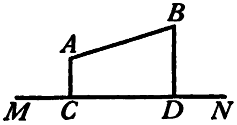
7. State and prove the converse of the theorem in § 190.

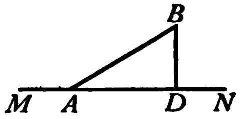
8. The sides of a triangle are a, b, and c. Derive formulæ for the segments into which the side c is divided by the bisector of the opposite angle.

9. Study the adjoining figure, and prove how a given line-segment AB may be divided into segments proportional to two given line-segments m and n, by application of § 189. ($AC = 2m$ and $BC = 2n$.)



191. Projection. — If from the extremities of a given line-segment perpendiculars are drawn to a given line, the segment cut off from the line by the perpendiculars is called the **projection** of the given line-segment upon the given line.

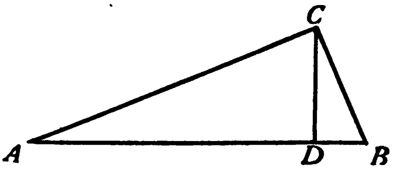




For example, if $AC \perp MN$ and $BD \perp MN$, the projection of AB upon MN is CD. If A is on MN, and $BD \perp MN$, as in the figure to the right, the projection of AB upon MN is AD. Why?

If the given line-segment AB intersects the line MN, draw a figure to show the projection of AB upon MN.

192. Theorem. — *Either leg of a right triangle is a mean proportional between the hypotenuse and its projection upon the hypotenuse.*



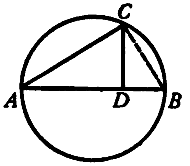
Hypothesis. In $\triangle ABC$, $\angle C$ is a right angle, and AD and DB are the projections of AC and BC, respectively, upon AB.

Conclusion. $\frac{AB}{AC} = \frac{AC}{AD}$ and $\frac{AB}{BC} = \frac{BC}{DB}$.

Suggestions. It may be proved that $\frac{AB}{AC} = \frac{AC}{AD}$ if what triangles are first proved similar? It may be proved that $\frac{AB}{BC} = \frac{BC}{DB}$ if what triangles are first proved similar?

Write out the complete proof.

193. Corollary. — *A chord of a circle is a mean proportional between the diameter drawn from one of its extremities and its projection upon that diameter.*



The proof is left to the student.

EXERCISES

1. In the figure of § 192, if $AB = 25$ in. and $AD = 16$ in., find AC and BC.

2. The hypotenuse of a right triangle is 18 in. and a leg is 15 in. Find the projection of the leg upon the hypotenuse.

3. In the figure of § 192, prove that $\overline{AC}^2 = AB \times AD$ and $\overline{BC}^2 = AB \times DB$.

4. In any right triangle, the product of the legs equals the product of the hypotenuse and the altitude to the hypotenuse.

SUGGESTION. — In § 192, prove $AC \times BC = AB \times CD$. This equality would follow from what proportion? The proportion follows from what similar triangles? Hence begin by proving the triangles similar.

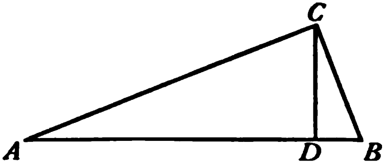
5. In any right triangle, the squares of the legs are to each other as their projections upon the hypotenuse.

SUGGESTION. — This may be proved by means of the equations in Ex. 3.

6. In any right triangle, the square of the hypotenuse and the square of either leg are proportional to the hypotenuse and the projection of that leg upon the hypotenuse.

SUGGESTION. — In the figure of § 192, how is the required proportion derived from the equations $\overline{AB}^2 = AB \times AB$ and $\overline{AC}^2 = AB \times AD$?

194. Theorem. — *In any right triangle, the altitude to the hypotenuse is the mean proportional between the two segments into which it divides the hypotenuse.*



Hypothesis. In $\triangle ABC$, $\angle C$ is a right angle, and $CD \perp AB$.

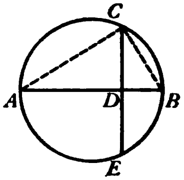
Conclusion. $\dfrac{AD}{CD} = \dfrac{CD}{DB}$.

Suggestion. This conclusion will follow if what triangles are proved similar?

Write the proof in full.

195. Corollary. — *The half of a chord perpendicular to a diameter is the mean proportional between the segments into which it divides the diameter.*

The proof is left to the student.



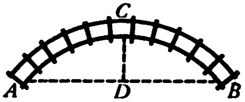
EXERCISES

1. The altitude to the hypotenuse of a right triangle divides it into segments of 12 in. and 27 in., respectively. Find the altitude to the hypotenuse.

2. The hypotenuse of a right triangle is 26 ft. The altitude to the hypotenuse is 12 ft. Find the segments into which the altitude divides the hypotenuse.

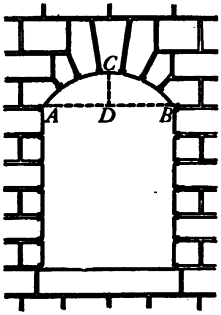
3. If one acute angle of a right triangle is 30°, the altitude to the hypotenuse divides the hypotenuse in the ratio of 1 to 3.

4. If one leg of a right triangle is twice the other, the altitude to the hypotenuse divides it in the ratio of 1 to 4.



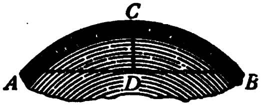
5. A railroad surveyor who wished to find the radius of the railroad curve ACB, measured the chord AB and the distance CD from the middle point of the arc to the middle point of the chord. Show how he was then able to compute the radius.

If $AB = 100$ ft. and $CD = 3$ ft., compute the radius of the curve.



6. A window whose width AB is 5 ft. is to be surmounted by a circular stone arch of which the rise CD is 20 in. Find the radius of the circle at which the stone for the arch must be cut.

7. In a bridge, a circular arch 18 ft. high is to span a stream 72 ft. wide. What is the radius of the circle at which the stones of this arch must be cut?

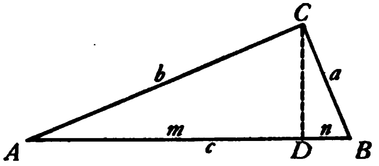


8. This is a piece of a broken wheel. $AB = 16$ in. and $CD = 4$ in. Find the diameter of the wheel.

9. The diameter of the earth may be computed as follows: Three stakes are set in a canal two miles long, one at each end and one in the middle, and all project the same distance above the water. By use of a leveling instrument the middle stake is found to be 8 inches higher than the others. From these facts compute the diameter of the earth.

10. If r is the radius of a circle, find the length of a chord whose distance from the center is $\frac{1}{2}r$.

196. Theorem. — *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*



Hypothesis. In $\triangle ABC$, $\angle C$ is a right angle. The legs are a and b, and the hypotenuse c.

Conclusion. $c^2 = a^2 + b^2$.

Proof. 1. In $\triangle ABC$, $\angle C$ is a right angle. Hyp.

2. Draw $CD \perp AB$, and let $AD = m$ and $DB = n$.

3. Then $\frac{c}{a} = \frac{a}{n}$ and $\frac{c}{b} = \frac{b}{m}$. § 192

4. $\therefore cn = a^2$ and $cm = b^2$. Ax. IV

5. $\therefore cn + cm = a^2 + b^2$. Ax. II

6. $\therefore c(n + m) = a^2 + b^2$. Factoring

7. $\therefore c \times c$ or $c^2 = a^2 + b^2$. Ax. XII

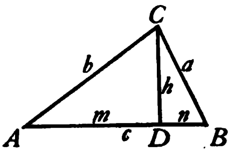
NOTE. — This theorem is called the *Pythagorean Theorem*, after the famous Greek philosopher and mathematician Pythagoras (about 540 B.C.), who is believed to have given the first rigorous proof of it.

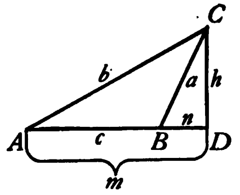
The truth of the theorem for special cases was known by the Egyptians centuries before the time of Pythagoras, and was used by them in building their pyramids. They laid out perpendicular lines by stretching a rope around three pegs so placed that the distances between them were proportional to 3, 4, and 5. The same principle is employed by carpenters and others now. Let the student, if possible, find some builder who is staking off the ground for the foundation of a house, and see how he obtains the right angles at the corners.

The proof above is attributed to the Hindus. Another proof of the theorem is given in § 231. It was given in the first great geometry, Euclid's *Elements*, about 300 B.C. Still other proofs are suggested in the exercises on page 229. Just what proof was given by Pythagoras is not known.

This is one of the most important and practical theorems of geometry.

197. Theorem. — *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of those sides and the projection of the other upon it.*





Hypothesis. — In $\triangle ABC$, $\angle A$ is acute, a, b, and c are the sides opposite A, B, and C, respectively, and m the projection of b upon c.

Conclusion. $a^2 = b^2 + c^2 - 2\,cm$.

Proof. 1. $\angle A$ is acute, a, b, and c are the sides opposite A, B, and C, respectively, and m is the projection of b upon c. Hyp.

2. Let n be the projection of a upon c, and let altitude $CD = h$.

Then if CD falls within $\triangle ABC$, $n = c - m$.

And if CD falls without $\triangle ABC$, $n = m - c$.

3. $\therefore$ squaring the members of either equation,

$$n^2 = m^2 + c^2 - 2\,cm. \qquad \text{Ax. VI}$$

4. Adding h^2 to each member,

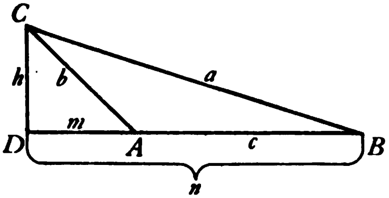
$$h^2 + n^2 = h^2 + m^2 + c^2 - 2\,cm. \qquad \text{Ax. II}$$

5. But $h^2 + n^2 = a^2$ and $h^2 + m^2 = b^2$. § 196

6. $\therefore a^2 = b^2 + c^2 - 2\,cm$. Ax. XII

If CD falls on CB, the triangle is a right triangle. Show that the theorem is true in that case.

198. Theorem. — *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of those sides and the projection of the other upon it.*



Hypothesis. In $\triangle ABC$, $\angle A$ is obtuse, a, b, and c are the sides opposite A, B, and C, respectively, and m the projection of b upon c.

Conclusion. $a^2 = b^2 + c^2 + 2\,cm$.

Suggestions. Since $n = m + c$, show that $n^2 = m^2 + c^2 + 2\,cm$. Then add h^2 to each member of the equation, and proceed as in the proof of § 197.

Write the proof in full.

199. Theorem. — *If a, b, and c are the sides opposite A, B, and C, respectively, of $\triangle ABC$, then:*

$\angle A$ *is acute if* $a^2 < b^2 + c^2$,

$\angle A$ *is obtuse if* $a^2 > b^2 + c^2$,

$\angle A$ *is a right angle if* $a^2 = b^2 + c^2$.

Suggestion. Each part of the theorem may be proved indirectly by use of § 196, § 197, and § 198.

Thus, for proving the first part, suppose that $\angle A$ is not acute. Then it is either obtuse or a right angle. If $\angle A$ is obtuse, $a^2 > b^2 + c^2$ by § 198; and if $\angle A$ is a right angle, $a^2 = b^2 + c^2$ by § 196. But these are untrue by hypothesis, etc.

Write the proof in full.

EXERCISES

1. The legs of a right triangle are 12 in. and 9 in. Find the hypotenuse.

2. The hypotenuse of a right triangle is 18 ft. and one leg is 14 ft. Compute the length of the other leg to tenths of a foot.

3. The sides of a triangle are 8 in., 9 in., and 12 in. What kind of angle is the largest angle of the triangle?

4. The sides of a triangle are 12 yd., 16 yd., and 20 yd. What kind of angle is the largest angle of the triangle?

5. Prove that a triangle whose sides are 3 ft., 4 ft., and 5 ft. is a right triangle.

6. Prove the principle used by the Egyptians, that a triangle whose sides are proportional to 3, 4, and 5 is a right triangle.

SUGGESTION. — Let $3\,n$, $4\,n$, and $5\,n$ be the three sides.

7. The legs of a right triangle are 8 and 12 respectively. Find the lengths of their projections upon the hypotenuse.

8. A baseball diamond is a square whose side is 90 ft. Find the length of the throw across from first to third base.

9. Find to hundredths of an inch the diagonal of a square whose side is 4 in.

10. Find a formula for the diagonal of a square whose side is s.

11. Find the altitude of an equilateral triangle if each side is 12 in.

12. If each of the equal sides of an isosceles trapezoid is 24 ft., one base 40 ft., and the other base 56 ft., find the altitude.

13. How long a rope is required to reach from the top of a tent pole 10 ft. high to a peg in the ground 12 ft. from the foot of the pole?

14. The gable of a house is to be made 18 ft. wide and 8 ft. high above the eaves. How long must the rafters be cut if they are to extend 18 in. below the eaves?

15. A piece of cloth 36 in. wide is to be cut on the bias, at an angle of 45° with the edge. How long will the bias cut be?

16. For reaming round holes 2 in. in diameter a square reamer is used. Find the dimensions of the reamer.

17. Tools and other pieces of metal often are made from stock in the form of round rods of various diameters. A tool is to be made with a square end whose side is $\frac{3}{4}$ in. What size (diameter) of stock must be selected?

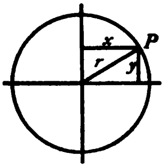
18. Show that if the sides of a triangle are numerically a, $\frac{1}{4}a^2 - 1$, and $\frac{1}{4}a^2 + 1$, it is a right triangle.

19. Show that $a^2 - b^2$, $a^2 + b^2$, and $2ab$ represent numerically the sides of a right triangle.

20. Pythagoras showed that $2n + 1$, $2n^2 + 2n$, and $2n^2 + 2n + 1$ were the sides numerically of a right triangle. Verify this.

21. This is an ancient Chinese problem: "A pool of water was 10 ft. across, and in the middle of it stood a reed which projected one foot above the water. When the wind blew the reed over, the top just reached to the edge of the pool. How deep was the water?"

22. Show that if two perpendicular lines are drawn through the center of a circle, and x and y are the perpendicular distances of any point P of the circle from these lines, and r is the radius, then $x^2 + y^2 = r^2$.



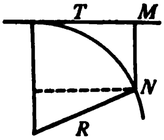
23. The diameters of two concentric circles are 20 in. and 36 in., respectively. Find the length of a chord of the larger that is tangent to the smaller.

24. The radius of a circle is 16 in. Find the length of a tangent to the circle from a point 24 in. from the center.

25. The radius of a circle is 28 in. Find to tenths of an inch the length of the shortest chord that can be drawn through a point 6 in. from the center.

26. The lengths of the radii of two circles are 10 in. and 6 in., respectively, and the distance between their centers 20 in. Find the lengths of their exterior common tangents.

27. Railroad curves are sometimes laid out by measuring offsets from the tangent line at the beginning of the curve. Show that if R is the radius of the curve and T the distance along the tangent from the beginning of the curve to any offset MN, then the offset $MN = R - \sqrt{R^2 - T^2}$.

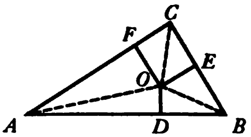


28. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides equals the sum of the squares of the other pair.

29. In $\triangle ABC$, $OD \perp AB$, $OE \perp BC$, $OF \perp AC$. Prove that

$$\overline{AD}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{DB}^2 + \overline{EC}^2 + \overline{FA}^2.$$

SUGGESTION. — $\overline{AD}^2 + \overline{DO}^2 = \overline{FA}^2 + \overline{FO}^2$, etc.

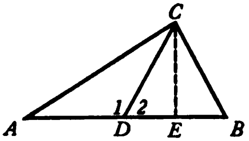


30. The sum of the squares of two sides of a triangle equals twice the square of one half of the third side plus twice the square of the median to that side.

HYPOTHESIS. — In $\triangle ABC$, CD is a median.

CONCLUSION. — $\overline{AC}^2+\overline{BC}^2=2\,\overline{AD}^2+2\,\overline{CD}^2$.

SUGGESTIONS. — Draw $CE \perp AB$. If $\angle 1$ is obtuse and $\angle 2$ acute, then $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + 2\,AD \times DE$ and $\overline{BC}^2 = \overline{DB}^2 + \overline{CD}^2 - 2\,DB \times DE$. Why?

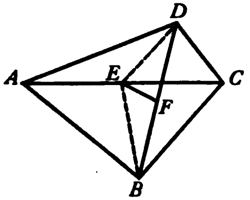


Add these equations and substitute AD for DB.

How does the proof differ if E falls between A and D? If E falls on D?

31. In any quadrilateral, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line-segment joining the middle points of the diagonals.

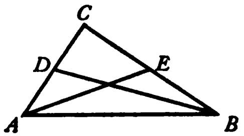
SUGGESTION. — ED, EB, and EF are medians of $\triangle ACD$, $\triangle ABC$, and $\triangle BDE$, respectively. Apply Ex. 30 to each of these triangles.



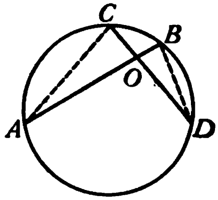
32. In any parallelogram the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.

33. If medians are drawn from the extremities of the hypotenuse of a right triangle, four times the sum of the squares of the medians is equal to five times the square of the hypotenuse.

To prove that $4(\overline{AE}^2 + \overline{BD}^2) = 5\,\overline{AB}^2$.



200. Theorem. — *If two chords intersect, the product of the segments of one is equal to the product of the segments of the other.*



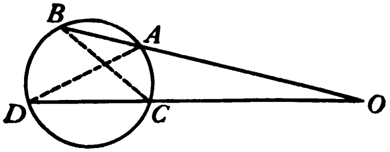
Hypothesis. Chords AB and CD intersect at a point O.

Conclusion. $AO \times OB = CO \times OD$.

Suggestion. Use proportion in similar triangles.

201. A secant and its external segment. — In the following theorems, by *a secant from an external point to a circle* is meant the segment of the secant between the given point and the farther point of intersection of the secant and circle. By the *external segment* of the secant is meant the segment of the secant between the given point and the nearer point of intersection of the secant and circle.

202. Theorem. — *If from a point without a circle two secants are drawn, the product of one secant and its external segment equals the product of the other secant and its external segment.*

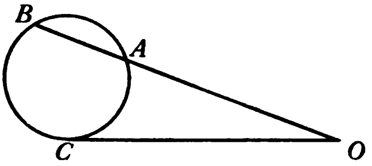


Hypothesis. Secants OB and OD intersect a circle at A and B, and C and D, respectively.

Conclusion. $OA \times OB = OC \times OD$.

Suggestion. Use proportion in similar triangles.

203. Theorem. — *If from a point without a circle a secant and a tangent are drawn, the product of the secant and its external segment equals the square of the tangent.*



Hypothesis. Secant OB intersects a circle at A and B, and OC is tangent at C.

Conclusion. $OA \times OB = \overline{OC}^2$.

Proof. The proof is left to the student.

EXERCISES

1. Two chords intersect so that the segments of one are 6 in. and 10 in. If the shorter segment of the other is 5 in., find the longer segment.

2. One of two intersecting chords is 8 in. long and the other is 12 in. long. If the segments of the first are 5 in. and 3 in., respectively, find the segments of the other.

3. In a circle whose radius is 12 in., a chord 18 in. long is drawn through a point 8 in. from the center. Find the segments into which the chord is divided at the point.

4. Two secants are drawn from an external point to a circle. One secant and its external segment are 14 in. and 6 in., respectively. If the other secant is 12 in., find its external segment.

5. A secant and a tangent are drawn from the same point to a circle. The chord intercepted on the secant is 4 in., and the external segment is 7 in. Find the length of the tangent.

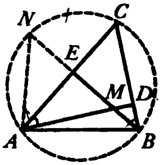
6. A tangent and a secant are drawn from a point to a circle, the secant passing through the center. If the tangent is 10 in. long and the external segment of the secant 4 in. long, find the radius.

7. If the radius of a circle is 12 in. and the length of a tangent from a point 15 in., find the lengths of the secant drawn from the point through the center and its external segment.

8. If a quadrilateral is inscribed in a circle, the product of the segments of one diagonal equals the product of the segments of the other diagonal.

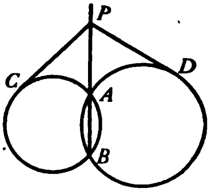
9. In $\triangle ABC$, altitudes AD and BE intersect at M. Prove that $AM \times MD = BM \times ME$.

SUGGESTION. — A circle may be drawn through A, B, D, and E. Why?



10. In $\triangle ABC$, altitudes AD and BE intersect at M. Prove that $BE \times EM = AE \times EC$.

SUGGESTION. — Circumscribe a circle about $\triangle ABC$, and produce BE to intersect it at N. Now $BE \times EN = AE \times EC$. Why? Hence it must be proved that $EM = EN$.

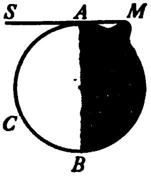


11. Tangents to two intersecting circles from any point on the common chord produced are equal.

12. If the common chord of two intersecting circles be produced to intersect their common tangent, it will bisect the tangent.

13. A bridge spans a stream 60 ft. wide, and the stone arch of the bridge is 20 ft. above the water at the center. Find the radius of the arch by use of § 200. This must be known before the arch can be constructed.

14. Galileo (1564–1642) measured the heights of the mountains on the moon as follows: ACB was the illuminated half of the moon just as the peak of the mountain M caught the beam SM of the rising or setting sun. He measured the distance AM from the half-moon's straight edge AB to the mountain peak M. Then by using the known diameter of the moon, show how he was able to compute the height of the mountain.



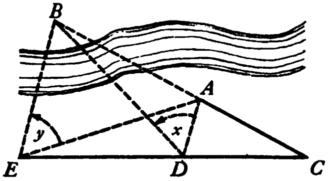
15. In the figure of Ex. 14, the known diameter AB of the moon is 2162 miles. If the distance AM is found to be 90 miles, how high is the mountain M?

16. Assuming the diameter of the earth to be 8000 mi., how far can one see on the surface of the earth from the top of a mountain 2 mi. high which rises out of a level plain?

17. The top of a mountain is visible from the surface of the water 60 mi. out at sea. How high is the mountain above the sea level?

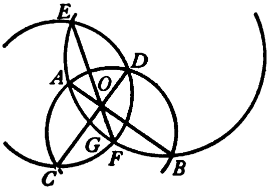
18. The light in a lighthouse 60 ft. above the water can be seen from the surface of the water how far away?

19. What is the greatest distance on the surface of the earth that a man can see from an aëroplane if he is one mile high?



20. A surveyor proceeded as follows to find the approximate distance from a point A to an inaccessible point B: He walked from A 100 steps to C in a line with A and B. Then he walked 100 steps to a convenient point D, and noted $\angle x$. He then continued in line CD until he reached a point E from which $\angle y = \angle x$. Show that $DE = AB$.

Suggestion. — It can be proved that $DE = AB$ if it is first proved that a circle may be drawn through the four points A, B, D, and E. See Exercise 29, page 180.



21. If three circles intersect one another, the three common chords all pass through the same point.

Suggestion. — Let the chords AB and CD intersect at O. Draw EO and produce it. Suppose that EO produced meets arc EAB again at F and arc EDC at G. Prove that $OF = OG$, and hence that F and G coincide.

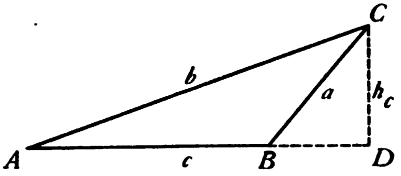
22. The line of centers of two circles intersects the circles at A and B and C and D, respectively, and their common external tangent at E. Prove that $AE \times DE = BE \times CE$.

Suggestion. — The required equation would follow from what proportion? For proving this proportion, draw chords from A and B and from C and D to the points of contact of the common tangent. Also draw the radii to the points of contact.

23. If the line of centers of two circles intersects the common external tangent at E, and any secant from E intersects the circles at A and B and C and D, respectively, show that $AE \times DE = BE \times CE$.

204. Theorem. — *If a, b, and c are the sides of a triangle, s one half of the perimeter, and h_c the altitude to c, then*

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$



Hypothesis. In $\triangle ABC$, a, b, and c are sides opposite A, B, and C, respectively, $s = \frac{1}{2}(a+b+c)$, and h_c the altitude to c.

Conclusion. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Proof. 1. Since one base angle must be acute, let it be $\angle A$.

2. Then $h_c^2 = b^2 - \overline{AD}^2$. § 196 and Ax. III

3. Also $a^2 = b^2 + c^2 - 2c \times AD$. § 197

4. Solving step 3, $AD = \frac{b^2+c^2-a^2}{2c}$.

5. Hence, substituting for AD in step 2,

$$h_c^2 = b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2. \qquad \text{Ax. XII}$$

6. ∴ reducing to a fraction and factoring,

$$h_c^2 = \frac{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}{4c^2}.$$

7. But $\frac{1}{2}(a+b+c) = s$, and hence $a+b+c = 2s$. Hyp.

8. ∴ $b+c-a = 2(s-a)$, $a-b+c = 2(s-b)$, $a+b-c = 2(s-c)$. Ax. III

9. ∴ $h_c^2 = \frac{4s}{c^2}(s-a)(s-b)(s-c)$. Ax. XII

10. ∴ $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. Ax. VI

EXERCISES

1. Show by interchanging letters in the proof of § 204 that
$h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$ and $h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$.

2. If the sides of a triangle are 8 in., 9 in., and 11 in., find s. Find $s-a$, $s-b$, and $s-c$.

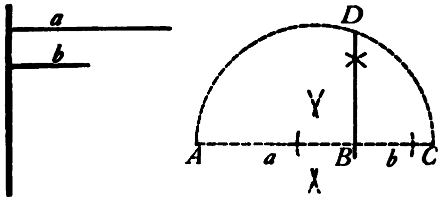
3. Find the three altitudes of a triangle whose sides are 5 in., 6 in., and 7 in.

4. Find the altitudes of a triangle whose sides are 20 ft., 24 ft., and and 30 ft.

5. If a is a side of an equilateral triangle, find the formula for an altitude by § 204.

6. If a side of an equilateral triangle is 8 ft., find an altitude.

205. Construction. — *Construct the mean proportional between two given line-segments.*



Given line-segments a and b.

Required to construct the mean proportional between a and b.

Construction. 1. On a straight line mark off $AB = a$ and $BC = b$.

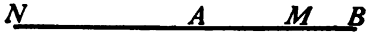
2. With AC as diameter, draw a semicircle.

3. Draw $BD \perp AC$, intersecting the semicircle at D.

4. Then BD is the required mean proportional.

Proof. The proof is left to the student. Write the construction and proof in full.

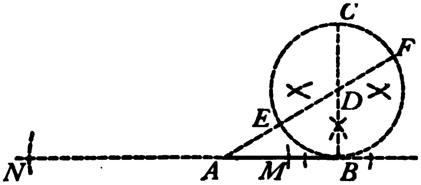
206. Extreme and mean ratio. — A line-segment is said to be divided in **extreme and mean ratio** by a point when one part is the mean proportional between the other part and the whole line-segment.



Thus, AB is divided by M *internally* in extreme and mean ratio if $\frac{AB}{AM}=\frac{AM}{MB}$, and by N *externally* in extreme and mean ratio if $\frac{AB}{AN}=\frac{AN}{NB}$.

The problem to divide a given line-segment in extreme and mean ratio has been called, since the time of the ancient Greeks, the *Problem of the Golden Section.*

207. Construction. — *Divide a given line-segment in extreme and mean ratio.*



Given line-segment AB.

Required to divide AB in extreme and mean ratio.

Construction. 1. Construct $BC \perp AB$, making $BC = AB$.

2. With BC as diameter, draw a circle.

3. Draw a line from A through D, the center, intersecting the circle at E and F.

4. On AB mark off $AM = AE$, and on AB produced through A mark off $AN = AF$.

5. Then AB is divided in extreme and mean ratio internally at M and externally at N.

Proof. 1. $BC \perp AB$. Construction

2. $\therefore AB$ is tangent to circle with diameter BC. § 163

3. $\therefore AE \times AF = \overline{AB}^2$. § 203

4. But $AE = AM$ and $EF = BC$ (diam.) $= AB$. Construction

5. $\therefore AM(AM + AB) = \overline{AB}^2$. Ax. XII

6. $\therefore \dfrac{AM + AB}{AB} = \dfrac{AB}{AM}$. Ax. V

7. $\therefore \dfrac{AM}{AB} = \dfrac{AB - AM}{AM}$ or $\dfrac{MB}{AM}$. § 118, (6)

8. $\therefore \dfrac{AB}{AM} = \dfrac{AM}{MB}$. § 118, (3)

By substituting $AN - AB$ for AE and AN for AF in step 3, then applying Ax. V and § 118, (5), let the student complete the proof that $\dfrac{AB}{AN} = \dfrac{AN}{NB}$.

Write the construction and proof in full.

EXERCISES

1. Draw a line-segment 1 in. long and one 4 in. long. Construct the mean proportional between them. Measure the resulting mean proportional. Is the construction accurate?

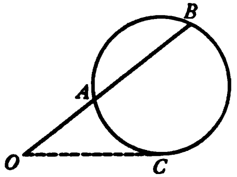
2. Divide a line-segment 4 in. long internally in extreme and mean ratio. Measure the parts to the nearest sixteenth of an inch. Is the construction accurate?

3. If a and b are two given line-segments, construct a line-segment n such that $n = \sqrt{ab}$.

4. Compute the parts of a line-segment 20 in. long when divided in extreme and mean ratio.

5. Experience has shown that a book, photograph, or other rectangular object is most pleasing to the eye when its length and width are obtained by dividing the half-perimeter in extreme and mean ratio. Find to the nearest integer the width of such a book whose length is 8 in.

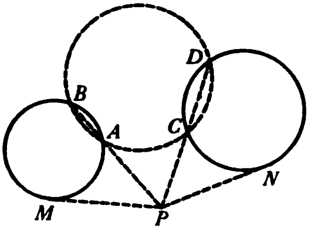
6. Measure the lengths and widths of several books, picture frames, or windows, and find whether or not the Golden Section is apparently used in their construction.



7. Given the longer part of a line-segment divided in extreme and mean ratio, construct the line-segment.

8. From a given external point O, draw a secant OB to intersect a given circle so that $\overline{AB}^2 = OA \times OB$.

9. Find a point P in the arc AB so that $\dfrac{PA}{PB} = \dfrac{3}{2}$.

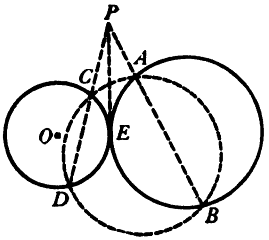


10. Find a point such that tangents from it to two given circles shall be equal.

11. Through one of the intersections of two given circles draw a line such that the two chords intercepted on it by the circles shall be in the ratio of 3 to 4.

12. Draw a circle which shall pass through two given points and be tangent to a given line.

Suggestion. — See § 203.



13. Draw a circle passing through two given points and tangent to a given circle.

Suggestion. — Let A and B be the given points and O the center of the given circle. Draw any circle through A and B, cutting the given circle at C and D. Draw AB and CD, and let them intersect at P. Draw PE tangent to the given circle. The required circle will be tangent to PE.

14. Draw a circle through a given point and tangent to two given lines.

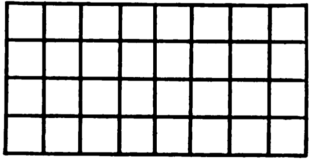
Suggestion. — Draw a line through the given point and perpendicular to the bisector of the angle formed by the given lines. On this line locate a second point at the same distance from the bisector as that of the given point. Then apply Ex. 12.

CHAPTER X

AREAS OF POLYGONS

208. Area of a surface. — The **area** of a surface is the *numerical measure* of the surface, the *unit of measure* being the surface of a square whose side is some linear unit.

Thus, in the adjoining figure, if the surface of one of the small squares is taken as the unit of measure, the *area* of the rectangle is 32.

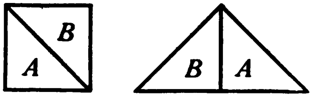


In the theorems and constructions involving areas of polygons, by the words "rectangle," "parallelogram," etc., are meant the *areas* of the figures mentioned.

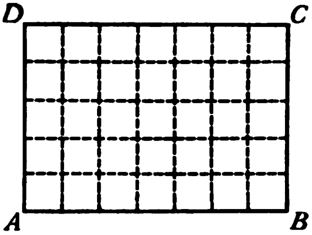
209. Equal polygons. — Two polygons are said to be **equal** when they have equal areas.

It is evident that two *congruent* polygons are *equal*, but that two equal polygons are not necessarily congruent.

Thus, in the adjoining figures, the square and isosceles triangle are equal, for the isosceles triangle is formed by dividing the square into two triangles, *A* and *B*, and placing them in new positions. But the square and isosceles triangle are not congruent.

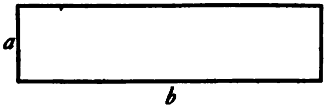


210. The area of a rectangle. — If the base *AB* of a rectangle *ABCD* is 7 linear units, and the altitude *AD* is 5 linear



units, the rectangle may be divided into 5 rows of squares with 7 squares in each row. Hence the total number of squares is 5 × 7 or 35.

In general, if the altitude of a rectangle is a linear units and the base is b of those units, the number of unit squares of surface, or the area of the rectangle, is ab.



This relation may be proved to be equally true when the altitude and base have no common unit of measure, or when their numerical measures can be expressed only approximately, although the proof is not attempted in this course. That is,

The area of a rectangle is equal to the product of its altitude and base.

211. Corollary 1. — *The area of a square is equal to the square of its side.*

For, if the side is a, the area equals $a \times a$ or a^2, by § 210.

212. Corollary 2. — *Two rectangles with equal altitudes are to each other as their bases.*

For, if a and b are the altitude and base, respectively, of rectangle M, and a and c are the altitude and base, respectively, of rectangle N, then by § 210,

$$M = ab \text{ and } N = ac.$$

Hence, by dividing, $\frac{M}{N} = \frac{ab}{ac}$ or $\frac{b}{c}$.

213. Corollary 3. — *Two rectangles with equal bases are to each other as their altitudes.*

The proof is left to the student.

EXERCISES

1. Find the area of a rectangle whose base is 8.45 in. and altitude 5.38 in.

2. Find the area of a square whose side is $6\frac{5}{8}$ in.

3. The area of a square is 160 sq. rd. Find the side of the square to hundredths of a rod.

4. All lots of a city block are 120 ft. long. If lot A in this block is 50 ft. wide and lot B is 60 ft. wide, compare the areas of the two lots (find their ratio in lowest terms).

5. Prove that any two rectangles are to each other as the products of their bases and altitudes.

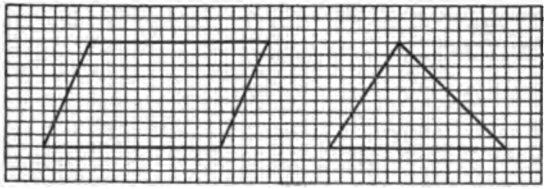
6. If one field is 80 rd. long and 40 rd. wide, and another field is 60 rd. square, find the ratio of their areas without computing them.

7. A rectangle is 50 in. wide and 200 in. long. Compare its perimeter with that of an equal square.

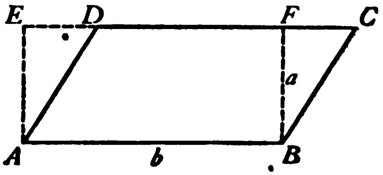
8. The perimeter of a rectangular lot is 220 yd. and its area 2400 sq. yd. Find the dimensions of the lot.

Suggestion. — Form an equation.

9. Find the approximate areas of the parallelogram and triangle below, using one of the small squares as the unit of area. In counting the squares, include a square in the figure if one half of it or more lies within the figure; otherwise do not include it.



214. Theorem. — *The area of a parallelogram is equal to the product of its altitude and base.*



Hypothesis. $ABCD$ is a parallelogram, with altitude a and base b.

Conclusion. The area of $\square\, ABCD = ab$.

Suggestions. Construct rectangle $ABFE$, with altitude a and base b. Prove that $\square\, ABCD =$ rect. $ABFE$, by proving $\triangle ADE \cong \triangle BCF$. Then apply § 210 and Ax. I.

Write the complete proof.

215. Corollary 1. —*Parallelograms having equal altitudes and equal bases are equal.*

The proof is left to the student.

216. Corollary 2. — *Two parallelograms having equal altitudes are to each other as their bases.*

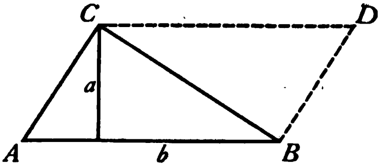
Suggestion. The proof is similar to that in § 212.

217. Corollary 3. — *Two parallelograms having equal bases are to each other as their altitudes.*

The proof is left to the student.

218. Base of a triangle. — Since every triangle has three altitudes, any side of a triangle may be considered as its **base** and the perpendicular to it from the opposite vertex as the corresponding altitude.

219. Theorem. — *The area of a triangle is equal to one half of the product of its altitude and base.*



Hypothesis. $\triangle ABC$ has altitude a and base b.

Conclusion. $\triangle ABC = \frac{1}{2} ab$.

Suggestions. Construct $\square\, ABDC$ with altitude a and base b. Then $\triangle ABC = \frac{1}{2} \square\, ABDC$. Why? Apply § 214.

220. Corollary 1. — *Triangles having equal altitudes and equal bases are equal.*

The proof is left to the student.

221. Corollary 2. — *Two triangles having equal altitudes are to each other as their bases.*

The proof is left to the student.

222. Corollary 3. — *Two triangles having equal bases are to each other as their altitudes.*

The proof is left to the student.

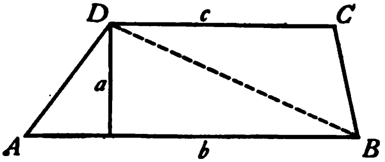
223. Corollary 4. — *A triangle is equal to one half of a parallelogram having the same base and altitude.*

This is shown in the proof of § 219.

224. Corollary 5. — *The area of a triangle whose sides are* a, b, *and* c *is equal to* $\sqrt{s(s-a)(s-b)(s-c)}$, *where* s *is one half of the perimeter.*

This follows from § 204 and § 219. Explain.

225. Theorem. — *The area of a trapezoid is equal to one half of the product of its altitude and the sum of its bases.*



Hypothesis. Trapezoid $ABCD$ has altitude a and bases b and c.

Conclusion. Trapezoid $ABCD = \frac{1}{2} a(b + c)$.

Suggestion. Draw DB. Prove that $\triangle ABD = \frac{1}{2} ab$ and $\triangle DCB = \frac{1}{2} ac$. Then add and factor.

226. Corollary. — *The area of a trapezoid is equal to the product of its altitude and the line-segment joining the middle points of its non-parallel sides.*

See § 98.

227. Area of any polygon. — The area of any polygon may be obtained by dividing it into triangles, parallelograms, and trapezoids, and then finding the area of each of these figures separately by use of the preceding theorems.

EXERCISES

1. Find the area of a parallelogram whose base is $12\frac{3}{4}$ in. and altitude $7\frac{1}{2}$ in.

2. Find the altitude of a parallelogram whose area is 100 sq. ft. and base $8\frac{1}{3}$ ft.

3. Find the area of a triangle whose base is 144 rd. and altitude 64 rd.

4. Find the base of a triangle whose area is 228 sq. ft. and altitude 12 ft. 8 in.

5. Find the area of a trapezoid whose bases are 17 in. and 23 in. and altitude 15 in.

6. The area of a trapezoid is 1701 sq. yd., the altitude is 42 yd., and one base 36 yd. Find the other base.

7. What is the area of a rhombus whose diagonals are 24 in. and 30 in., respectively?

8. Find the side of a square whose area equals that of a triangle whose base is 48 in. and altitude 24 in.

9. Find, to tenths, the area of an equilateral triangle whose side is 6 in.

10. Find the area of a triangle whose sides are 6 in., 8 in., and 12 in.

11. Find the area of a triangle whose sides are 15 ft., 16 ft., and 17 ft.

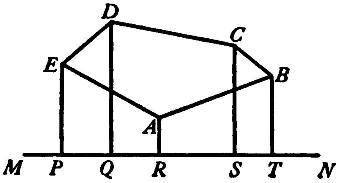
12. How many pieces of sod will it take to sod a lawn 42 ft. wide and 56 ft. long, if the pieces are 12 in. by 14 in.?

13. On a map drawn to the scale of 60 mi. to the inch, what area is inclosed in a strip 3 in. wide and 5 in. long?

14. A canal is being excavated that is 28 ft. deep, 120 ft. wide at the top, and 90 ft. wide at the bottom. What is the area of a cross section of it?

15. Surveyors sometimes find the area of a tract of land as follows: A base line MN is staked off north and south, say. From the various points in the boundary, the distances to this line are then measured, as EP, DQ, etc., and the distances PQ, QS, etc., are measured. Show how to compute the area of $ABCDE$.

If $EP = 2000$ ft., $DQ = 3000$ ft., $CS = 2700$ ft., $BT = 1500$ ft., $AR = 800$ ft., $PQ = 800$ ft., $QS = 2100$ ft., $ST = 800$ ft., $TR = 2000$ ft., compute the area.



16. Find, without multiplying, the ratio of the areas of two parallelograms whose bases are 10 in. and 16 in., and altitudes 8 in. and 12 in., respectively.

17. Construct a parallelogram which shall be twice a given parallelogram.

18. Construct a rectangle which shall equal one half of a given parallelogram.

19. By means of § 224 prove that the area of an equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$. Find the area if $s = 20$ in.

20. Prove that the diagonals of a parallelogram divide it into four equal triangles.

21. The parallelogram formed by joining the middle points of the adjacent sides of any quadrilateral is equal to one half of the quadrilateral.

22. If a point within a parallelogram is joined to the four vertices, the sum of the two triangles thus formed having for bases two parallel sides is equal to one half of the parallelogram.

23. Discuss Ex. 22 when the given point is on one of the sides.

24. Is the theorem in Ex. 22 true when the given point is without the given parallelogram? If not, what relation does exist between the triangles thus formed and the parallelogram? Prove the answer.

25. A parallelogram is bisected by any straight line drawn through the intersection of the diagonals.

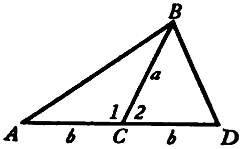
26. Any trapezoid is bisected by the line-segment joining the middle points of its bases.

27. If $ABCD$ is any trapezoid, and the diagonals AC and BD intersect at O, then $\triangle AOD = \triangle BOC$.

28. The area of a rhombus is equal to one half the product of its diagonals.

29. Two triangles are equal if two sides of one are equal to two sides of the other, and the included angles supplementary.

SUGGESTION. — Place the triangles with the supplementary angles adjacent, as in the figure.



30. The line-segments joining the middle points of the sides of any triangle divide the triangle into four equal triangles.

31. The triangle having one of the non-parallel sides of a trapezoid as base and the middle point of the opposite side as vertex is equal to one half of the trapezoid.

32. The area of a trapezoid is equal to the product of one of its non-parallel sides and the distance to it from the middle point of the opposite side.

33. If from the middle point of either diagonal of any quadrilateral line-segments are drawn to the two opposite vertices, they divide the quadrilateral into two equal quadrilaterals.

34. If from the point of intersection of the medians of any triangle line-segments are drawn to the three vertices, they form with the sides three equal triangles.

35. If from the middle point of any side of a triangle straight lines are drawn parallel to the other two sides, the parallelogram thus formed is equal to one half of the triangle.

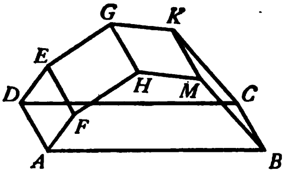
36. Two isosceles triangles having the altitude of one equal to one half of the base of the other, and their sides equal each to each, are equal.

37. If the middle points of the adjacent sides of any parallelogram are joined, the four triangles thus formed are equal, and together they are equal to the parallelogram thus formed by the line-segments joining the middle points.

38. The area of a triangle equals one half of the product of its perimeter and the radius of the inscribed circle.

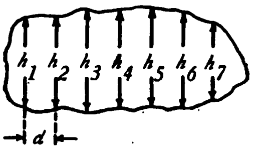
39. The papyrus of Ahmes, the oldest mathematical book extant, written by the Egyptian about 1700 B.C., gave for an isosceles triangle whose equal sides are 10 *ruths* and base 4 *ruths*, the area of 20 square ruths. By what rule must this area have been computed? What is the true area?

40. One base of a trapezoid is 10 ft., the altitude 4 ft., and the area 32 sq. ft. Find the length of the line-segment drawn between the non-parallel sides, parallel to the given base and 1 ft. from it.

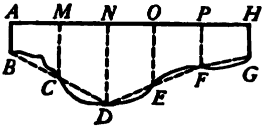


41. Prove that $\square AFED + \square FHGE + \square HMKG + \square MBCK = \square ABCD$.

42. A fairly accurate method used for computing the area bounded by an irregular curve, known as the *mean ordinate* method, is as follows: At equal distances d measure the widths h_1, h_2, h_3, etc., of the area inclosed. Show that if a large number of widths are measured, a close approximation to the area is given by $d(h_1 + h_2 + h_3 + \cdots)$.



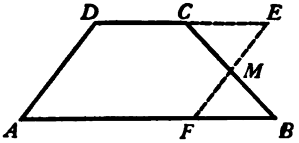
43. In order to determine the flow of water in a stream, the area of a cross section *ABCDEFGH* of the stream, at right angles to the current, is first found as follows: Soundings are taken at *M*, *N*, etc., then the areas of trapezoids *ABCM*, *MCDN*, etc., are computed and added.



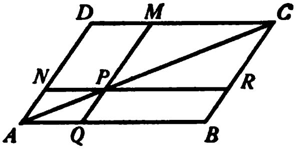
If $AM = MN = NO = OP = PH = 8$ ft., $AB = 5$ ft., $MC = 9$ ft., $ND = 13$ ft., $OE = 11$ ft., $PF = 7$ ft., and $HG = 5$ ft., find the area of the cross section.

This is known as the *trapezoidal rule* for finding an area. It is used in measuring the area of land bounded on one side by an irregular line.

44. If *M* is the middle point of side *BC* of trapezoid *ABCD*, and *EF* is drawn through *M*, parallel to *AD*, to meet *AB* at *F* and *DC* produced at *E*, prove that trapezoid $ABCD = \square AFED$.

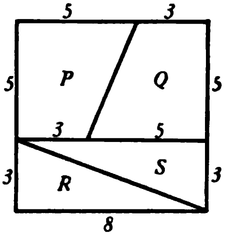


45. Through any point *P* on the diagonal *AC* of parallelogram *ABCD*, parallels are drawn to the four sides, forming parallelograms *NPMD* and *QBRP*. Prove $\square NPMD = \square QBRP$.

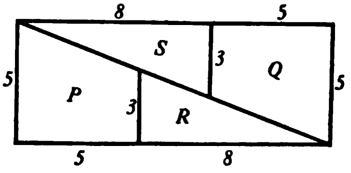


46. If *P* is any point on diagonal *AC* produced of parallelogram *ABCD*, then $\triangle PCD = \triangle PCB$.

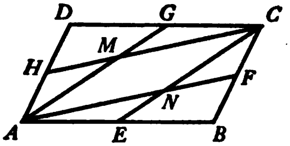
47. Explain the fallacy in the following puzzle: A piece of paper 8 in. square is cut into four pieces, *P*, *Q*, *R*, *S*, as shown in the figure. These pieces are then placed in new positions so as to form, apparently, a rectangle whose area is 65 sq. in.



SUGGESTION. — Use similar triangles. If the triangles fitted together so that their sides formed a true diagonal of a rectangle, what proportion would follow?



48. If from two opposite vertices of any parallelogram line-segments are drawn to the middle points of the sides, the parallelogram thus formed is equal to one third of the given parallelogram. That is, $\square ANCM = \frac{1}{3} \square ABCD$.



Suggestions. — First prove that quadrilateral *ANCM* is a parallelogram.

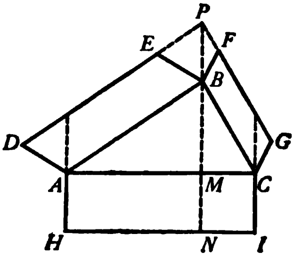
Then compare the bases and altitudes of $\square ANCM$ and $\square AECG$. For this purpose, draw diagonal *AC*, and apply the theorem in § 114 to $\triangle ABC$.

Hence $\square ANCM$ equals what part of $\square AECG$?

Similarly, $\square AECG$ equals what part of $\square ABCD$?

49. Upon two sides of any triangle *ABC* the parallelograms *ABED* and *BCGF* are drawn. *DE* and *GF* are produced to intersect at *P*, and *PB* is drawn. Parallelogram *AHIC* is drawn with side *AH* equal and parallel to *PB*. Prove that

$$\square AHIC = \square ABED + \square BCGF.$$



Suggestion. — Show that

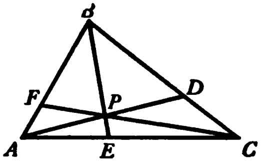
$$\square ABED = \square AHNM$$

and $$\square BCGF = \square MNIC.$$

For proving that $\square ABED = \square AHNM$, find the relation of each of these parallelograms to a third parallelogram.

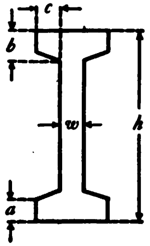
50. The lines *AD*, *BE*, and *CF* are drawn from the vertices of $\triangle ABC$ so that they meet at a point *P*. Show that

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$$



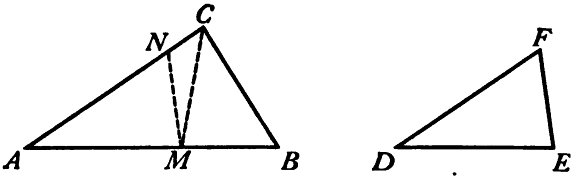
Suggestion. — $\frac{BD}{DC} = \frac{\triangle APB}{\triangle APC}$;

$\frac{CE}{EA} = \frac{\triangle BPC}{\triangle APB}$; $\frac{AF}{FB} = \frac{\triangle APC}{\triangle BPC}$.



51. The formula Area $= wh + 2c(a + b)$ for computing the area of the cross section of an I beam, where *w*, *h*, *c*, *a*, and *b* are the dimensions shown in the drawing, is given in books on engineering. Prove this formula.

228. Theorem. — *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$.

Conclusion. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \times AC}{DE \times DF}$.

Proof. 1. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$. Hyp.

2. $\therefore$ $\triangle DEF$ may be placed upon $\triangle ABC$ so that $\angle A$ and $\angle D$ coincide, becoming $\triangle AMN$. § 13

3. Draw MC.

4. Then $\dfrac{\triangle ABC}{\triangle AMC} = \dfrac{AB}{AM}$ and $\dfrac{\triangle AMC}{\triangle AMN} = \dfrac{AC}{AN}$. § 221

5. $\therefore$ $\dfrac{\triangle ABC}{\triangle AMC} \times \dfrac{\triangle AMC}{\triangle AMN} = \dfrac{AB}{AM} \times \dfrac{AC}{AN}$,

or $\dfrac{\triangle ABC}{\triangle AMN} = \dfrac{AB \times AC}{AM \times AN}$. Ax. IV

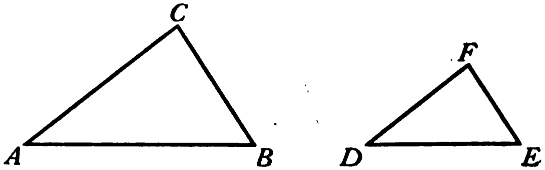
6. $\therefore$ $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \times AC}{DE \times DF}$. Ax. XII

EXERCISES

1. Two triangles having an angle of one supplementary to an angle of the other are to each other as the products of the sides including those angles.

2. Two parallelograms with equal angles are to each other as the products of the sides including a pair of equal angles.

229. Theorem. — *Two similar triangles are to each other as the squares of any two corresponding sides.*



Hypothesis. $\triangle ABC \sim \triangle DEF.$

Conclusion. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2}.$

Proof. 1. $\triangle ABC \sim \triangle DEF.$ Hyp.

2. $\therefore \angle A = \angle D.$ Def. sim. $\triangle$s

3. $\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \times AC}{DE \times DF} = \dfrac{AB}{DE} \times \dfrac{AC}{DF}.$ § 228

4. But $\dfrac{AB}{DE} = \dfrac{AC}{DF}.$ Def. sim. $\triangle$s

5. $\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE} \times \dfrac{AB}{DE} = \dfrac{\overline{AB}^2}{\overline{DE}^2}.$ Ax. XII

EXERCISES

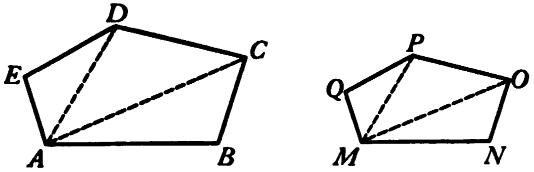
1. Two triangles that have an angle of one equal to an angle of the other, have the sides including the equal angles 4 in. and 9 in. and 12 in. and 5 in., respectively. Compare their areas.

2. Two corresponding sides of similar triangles are 8 in. and 12 in., respectively. If the area of the smaller triangle is 48 sq. in., find the area of the larger.

3. $\triangle ABC \sim \triangle DEF$ and AB and DE are corresponding sides. If $AB = 2\,DE$, compare the areas. If $AB = 3\,DE$, compare the areas.

4. The sides of a triangle are 3 in., 4 in., and 6 in. Find the sides of a similar triangle whose area is 9 times as great.

230. Theorem. — *Two similar polygons are to each other as the squares of any two corresponding sides.*



Hypothesis. $ABCD \cdots \sim MNOP \cdots$, and AB and MN are corresponding sides.

Conclusion. $\dfrac{ABCD \cdots}{MNOP \cdots} = \dfrac{\overline{AB}^2}{\overline{MN}^2}.$

Proof. 1. $ABCD \cdots \sim MNOP \cdots$, and AB and MN are corresponding sides. Hyp.

2. Draw all diagonals from A and from M.

3. Then $\triangle ABC \sim \triangle MNO$, $\triangle ACD \sim \triangle MOP$, etc. § 137

4. $\therefore \dfrac{\triangle ABC}{\triangle MNO} = \dfrac{\overline{AB}^2}{\overline{MN}^2}$, $\dfrac{\triangle ACD}{\triangle MOP} = \dfrac{\overline{CD}^2}{\overline{OP}^2}$, etc. § 229

5. But $\dfrac{AB}{MN} = \dfrac{CD}{OP} =$ etc. Def. sim. $\triangle$s

6. $\therefore \dfrac{\overline{AB}^2}{\overline{MN}^2} = \dfrac{\overline{CD}^2}{\overline{OP}^2} =$ etc. Ax. VI

7. $\therefore \dfrac{\triangle ABC}{\triangle MNO} = \dfrac{\triangle ACD}{\triangle MOP} =$ etc. Ax. I

8. $\therefore \dfrac{\triangle ABC + \triangle ACD + \text{etc}}{\triangle MNO + \triangle MOP + \text{etc.}} = \dfrac{\triangle ABC}{\triangle MNO}.$ § 118, (8)

9. $\therefore \dfrac{ABCD \cdots}{MNOP \cdots} = \dfrac{\triangle ABC}{\triangle MNO}.$ Ax. XII

10. $\therefore \dfrac{ABCD \cdots}{MNOP \cdots} = \dfrac{\overline{AB}^2}{\overline{MN}^2}.$ Ax. I

EXERCISES

1. Two corresponding sides of two similar polygons are 5 in. and 8 in., respectively, and the area of the smaller polygon is 125 sq. in. What is the area of the larger?

2. The sides of a quadrilateral are 4 in., 6 in., 10 in., and 12 in., respectively. Find the sides of a similar quadrilateral whose area is 16 times as great.

3. The adjacent sides of a parallelogram are 4 ft. and 12 ft., respectively. Find the corresponding sides of a similar parallelogram whose area is one fourth as great.

4. The areas of two similar polygons are 324 sq. ft. and 576 sq. ft., respectively. Compare their corresponding sides.

5. The area of a triangle is 64 sq. in. Lines parallel to the base divide one side into four equal segments. Find the areas of the four parts into which the triangle is divided.

6. If a line parallel to one side of a triangle divides another side into segments with the ratio of 2 to 3, what is the ratio of the areas of the two parts into which the triangle is divided?

7. A line parallel to the base AB of $\triangle ABC$ and dividing $\triangle ABC$ into two equal parts cuts off what part of AC, measured from C? Compute to hundredths.

8. Prove that the areas of two similar triangles have the same ratio as the squares of corresponding altitudes.

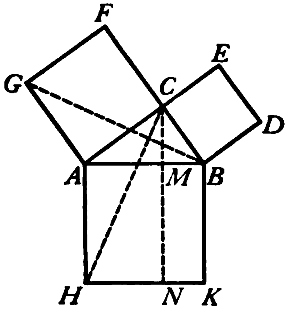
9. Prove that the areas of two similar polygons have the same ratio as the squares of corresponding diagonals.

10. Given a quadrilateral $ABCD$, construct a similar quadrilateral whose area shall be one fourth as great as that of $ABCD$.

11. If line-segments are drawn connecting the middle points of the segments of the diagonals of a given trapezoid, compare the area of the trapezoid thus formed with the area of the given trapezoid.

12. The illustrations of this book are printed from zinc etchings. The cost of an etching is computed at a certain price per square inch of its printing surface. If the illustrations had been made similar, but with the dimensions twice as great, how would the cost of the etchings have compared with the actual cost? If they had been made with their dimensions only three fourths as great, compare the cost of the etchings with the actual cost.

231. Theorem. — *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.*



Hypothesis. In right △ *ABC*, *AHKB* is the square on the hypotenuse, and *ACFG* and *CBDE* are squares on the legs.

Conclusion. Sq. *AHKB* = sq. *ACFG* + sq. *CBDE*.

Proof. 1. *AHKB*, *ACFG*, and *CBDE* are squares and ∠ *ACB* a right angle. Hyp.

2. Draw *GB* and *HC*, and draw *CN* ∥ *AH*, forming rect. *AHNM* and rect. *MNKB*.

3. ∠ *HAB* and ∠ *CAG* are rt. ∠s, and hence equal. Def. square

4. ∴ ∠ *HAC* = ∠ *BAG*. Ax. II

5. *AC* = *AG* and *AH* = *AB*. Def. square

6. ∴ △ *AHC* ≅ △ *ABG*. § 63

7. ∠ *FCA* = rt. ∠. Def. square

8. ∴ ∠ *FCB* = st. ∠. Def. rt. ∠

9. ∴ *FCB* is a straight line. Def. st. ∠

10. △ *AHC* = $\frac{1}{2}$ rect. *AHNM*; △ *ABG* = $\frac{1}{2}$ sq. *ACFG*. § 223

11. ∴ rect. *AHNM* = sq. *ACFG*. Ax. IV

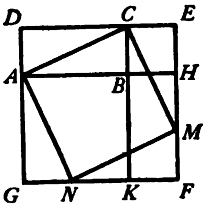
12. Similarly, rect. *MNKB* = sq. *CBDE*.

13. ∴ sq. *AHKB* = sq. *ACFG* + sq. *CBDE*. Ax. II

NOTE. — See note, § 196.

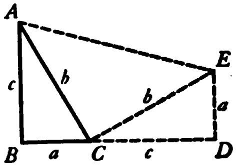
EXERCISES

1. Prove the theorem of § 231 by using the adjoining figure.



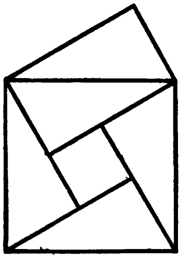
Suggestions. — $\triangle ABC$ is the right triangle, $ANMC$ the square on the hypotenuse, and $AGKB$ and $BHEC$ squares on the legs. If rect. $ABCD$ and rect. $BKFH$ are taken from sq. $GFED$, what remains? If the four congruent triangles at the corners are taken away, what remains?

2. James A. Garfield, once President of the United States, gave an original proof of the theorem in § 231. Given $\triangle ABC$, with $\angle B =$ rt. $\angle$, he constructed $\triangle CDE \cong \triangle ABC$ so that BCD was a straight line, and drew AE. He proved $\triangle ACE$ a right triangle, with $\angle ACE =$ rt. $\angle$. He then expressed the areas of $\triangle ABC$, $\triangle CDE$, and $\triangle ACE$ in terms of a, b, and c, and expressed the area of trapezoid $ABDE$. By putting the area of the trapezoid equal to the sum of the areas of the triangles, he obtained an equation from which the conclusion followed.

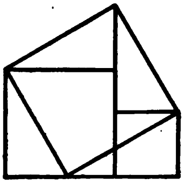


Write out his complete proof.

3. Bhaskara, a native of India about 1150 A.D., proved the Pythagorean Theorem by aid of a figure like that adjoining. Find his proof.



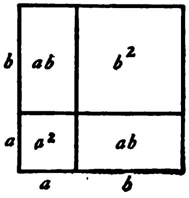
4. Still another proof of the Pythagorean Theorem may be obtained by aid of this figure. Give the proof.

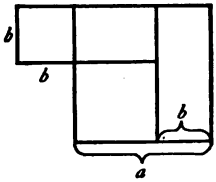


Suggestion. — The entire figure is composed of the square on the hypotenuse of the right triangle plus two triangles. It also is composed of the sum of the squares on the legs plus two congruent triangles.

5. The square on the sum of two line-segments is equal to the sum of the squares on the line-segments plus twice the rectangle formed by the line-segments as adjacent sides. That is,

$$(a + b)^2 = a^2 + b^2 + 2ab.$$

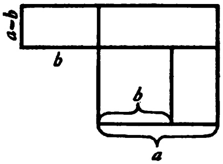




6. The square on the difference of two line-segments is equal to the sum of the squares on the line-segments minus twice the rectangle formed by the line-segments as adjacent sides. That is,

$$(a-b)^2 = a^2 + b^2 - 2\,ab.$$

Suggestion. — The entire figure is composed of the sum of the square on $a-b$ and two rectangles the dimensions of each of which are a and b. It also is composed of the sum of the squares on a and b.



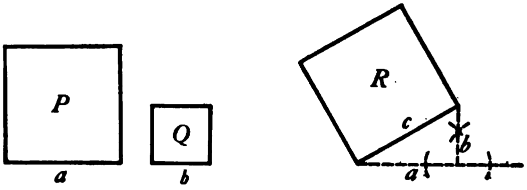
7. The rectangle formed by the sum and the difference of two line-segments as adjacent sides is equal to the difference between the squares on the two line-segments. That is,

$$(a+b)(a-b) = a^2 - b^2.$$

8. In any rectangle, the sum of the squares on the line-segments drawn from any point within the rectangle to two opposite vertices is equal to the sum of the squares on the line-segments drawn to the other two vertices.

Suggestion. — Through the point draw lines parallel to the sides.

232. Construction. — *Construct a square equal to the sum of two given squares.*



Given squares P and Q, with sides a and b, respectively.

Required to construct a square equal to $P+Q$.

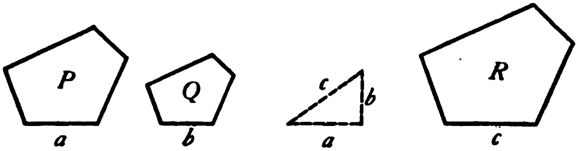
Construction. 1. Construct a right triangle with legs equal to a and b, respectively. Let the hypotenuse be c.

2. Upon c as a side, construct a square R.

3. Then R is the required square.

Proof. The proof is to be given by the student.

233. Construction. — *Construct a polygon similar to each of two given similar polygons and equal to their sum.*



Given two similar polygons P and Q, with corresponding sides a and b, respectively.

Required to construct a polygon similar to P and Q and equal to $P + Q$.

Construction. 1. Construct a right triangle with legs equal to a and b, respectively, and let the hypotenuse be c.

2. On c as a side corresponding to a, construct polygon $R \sim P$ and Q. § 139

3. Then R is the required polygon.

Proof. 1. $R \sim P$ and Q. Construction

2. c is hypotenuse of rt. △ with legs equal to a and b. Construction

3. $\therefore c^2 = a^2 + b^2$. § 231

4. $\frac{P}{R} = \frac{a^2}{c^2}$ and $\frac{Q}{R} = \frac{b^2}{c^2}$. § 230

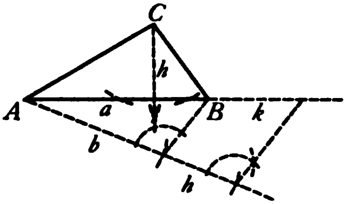
5. $\therefore \frac{P + Q}{R} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2}$ or 1. Ax. II and Ax. XII

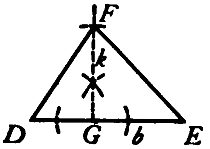
6. $\therefore P + Q = R$. Ax. IV

EXERCISE

1. If similar polygons are constructed on the sides of a right triangle as corresponding sides, the polygon on the hypotenuse is equal to the sum of the polygons on the two legs.

234. Construction. — *Construct a triangle on a given base and equal to a given triangle.*





Given △ ABC with base a, and line-segment b.

Required to construct a triangle on b as base and equal to △ ABC.

Construction. 1. Construct k, the fourth proportional to b, a, and h. § 126

2. Draw FG perpendicular to b and equal to k.
3. Join F to D and to E, the extremities of b.
4. Then △ DEF is the required triangle.

Proof. 1. $\frac{b}{a}=\frac{h}{k}$. Also b is base of △ DEF. Construction

2. $\therefore \frac{1}{2}kb = \frac{1}{2}ha$. Ax. IV
3. But $\frac{1}{2}kb =$ △ DEF and $\frac{1}{2}ha =$ △ ABC. § 219
4. $\therefore$ △ $DEF =$ △ ABC. Ax. I

235. Construction. — *Construct a triangle on a given base and equal to a given parallelogram.*

Analysis. If a is the base and h the altitude of the given parallelogram, and b the given base and k the altitude of the required triangle, then $ha = \frac{1}{2}kb$. Why? Hence $\frac{b}{2a}=\frac{h}{k}$. Therefore k must be constructed as the fourth proportional to b, $2a$, and h.

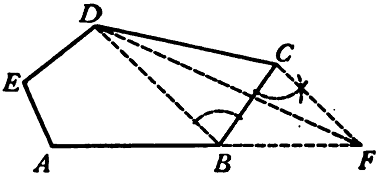
Make the construction. Write the construction and proof.

236. Construction. — *Construct a square equal to a given triangle.*

Analysis. If a is the altitude and b the base of the given triangle, and s a side of the required square, then $\frac{1}{2}ab = s^2$. Why? Show from this that $\frac{\frac{1}{2}a}{s} = \frac{s}{b}$. Hence s must be constructed as the mean proportional between $\frac{1}{2}a$ and b.

Make the construction. Write the construction and proof.

237. Construction. — *Construct a triangle equal to any given polygon.*



Given polygon $ABCD \cdots$.

Required to construct a triangle equal to $ABCD \cdots$.

Construction. 1. Draw diagonal DB.

2. Draw $CF \parallel DB$, and produce AB to intersect it at F.

3. Draw DF.

4. Then $AFD \cdots = ABCD \cdots$, and has one side less than $ABCD \cdots$.

5. By applying this construction in turn to the new polygon $AFD \cdots$, and continuing the process, finally a triangle will be obtained equal to $ABCD \cdots$.

Suggestion. For proving that $AFD \cdots = ABCD \cdots$, first prove that $\triangle DBF = \triangle DBC$, then add each triangle to polygon $ABD \cdots$.

Make the construction. Write the construction and proof.

238. Construction. — *Construct a square equal to any given polygon.*

Suggestion. First apply § 237, then § 236.

Make the construction. Write the construction and proof.

EXERCISES

1. Construct a square equal to the sum of any number of given squares.

2. Construct a square equal to the difference between two given squares.

3. Construct a square equal to twice a given square. Equal to three times a given square.

4. Construct a square equal to the sum of two given polygons.

SUGGESTION. — First construct squares equal to the given polygons.

5. Construct a polygon similar to each of three given similar polygons and equal to their sum.

6. Construct an isosceles triangle equal to a given triangle and on the same base.

7. Construct a square equal to a given parallelogram.

8. Construct a right triangle having a given altitude and equal to a given parallelogram.

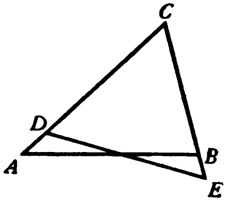
9. Construct a triangle having a given angle and equal to a given parallelogram.

10. Construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

11. Construct a rectangle having a given side and equal to a given square; equal to a given rectangle; equal to a given parallelogram.

12. Construct an isosceles triangle whose vertical angle is one of the angles of a given triangle, and whose area is equal to that of the given triangle.

SUGGESTION. — If $\triangle ABC$ is the given triangle and $\triangle CDE$ the required isosceles triangle, $\overline{CD}^2 = CA \times CB$.



13. Construct a triangle equal to a given trapezoid and having for its base the longer base of the trapezoid.

14. Construct a parallelogram having a given altitude and equal to a given trapezoid.

15. Construct a square equal to a given trapezoid.

16. Construct a rhombus having a given diagonal and equal to a given parallelogram.

17. Construct a triangle equal to a given triangle and having two of its sides equal to two given line-segments a and b.

18. Construct a polygon similar to one of two given polygons and equal to the other.

ANALYSIS. — If it is required to construct a polygon similar to P and equal to Q, let a and b be sides of squares equal to P and Q, respectively. Let R be the required polygon, and r a side of R corresponding to side p of P. Then $\frac{P}{R}=\frac{p^2}{r^2}$. Why? Also $\frac{P}{Q}=\frac{a^2}{b^2}$. Why? Hence, since $Q = R$, $\frac{a^2}{b^2}=\frac{p^2}{r^2}$. Why? Now show that r is the fourth proportional to a, b, and p.

19. Construct an equilateral triangle equal to a given triangle.

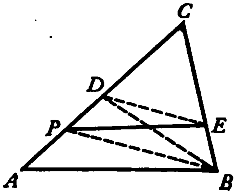
20. Construct an equilateral triangle equal to a given polygon.

21. Through a given point draw a straight line that shall bisect a given parallelogram.

22. Draw a straight line parallel to a given straight line and bisecting a given parallelogram.

23. Bisect a given triangle by a straight line drawn through any vertex.

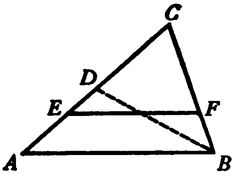
24. Divide a triangle into any given ratio by a straight line drawn through any vertex.



25. Bisect a triangle by drawing a straight line through any given point on a side of the triangle. Such a problem is sometimes encountered in land surveying.

ANALYSIS. — Let P be the given point. Since a line-segment from B to D, the middle point of AC, bisects the triangle, it follows that a point E must be found on BC such that $\triangle PEB = \triangle PDB$.

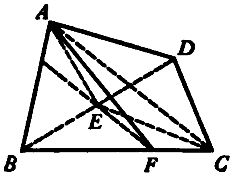
26. Trisect a given triangle by drawing straight lines through any given point on one of the sides.



27. Bisect a given triangle by a straight line drawn parallel to one of the sides.

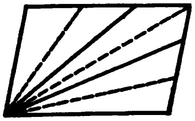
Analysis. — If EF is the required line, and DB a median, $\triangle DCB$ must equal $\triangle ECF$. Why? Hence $CE \times CF = CD \times CB$. Why? Consequently, $\frac{CD}{CE}=\frac{CF}{CB}$. Also $\frac{CE}{CA}=\frac{CF}{CB}$. Why? Therefore, $\frac{CD}{CE}=\frac{CE}{CA}$, that is, CE is the mean proportional between CD and CA.

28. Bisect any quadrilateral by a straight line drawn through one of the vertices.



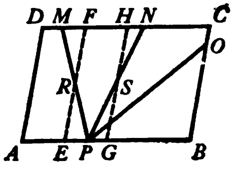
Analysis. — Let A be the vertex. If E is the middle point of BD, then AE and CE divide the quadrilateral into two equal parts. Hence it remains to construct a triangle equal to $\triangle AEC$, having for base AC and a vertex in one of the sides of the quadrilateral.

29. Trisect any given quadrilateral by straight lines drawn through any vertex.



30. Divide a given parallelogram into any required number of equal parts by straight lines drawn from one of its vertices.

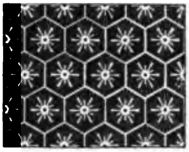
31. Divide a given parallelogram into four equal parts by drawing straight lines through any given point on one of its sides.



Suggestion. — Let P be the given point. Divide AB into four equal parts. Draw EF and GH parallel to AD. Bisect EF at R and GH at S. Having drawn PM and PN, construct PO by Ex. 28.

CHAPTER XI

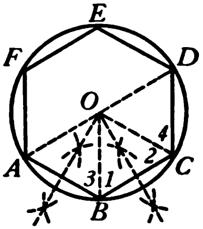
REGULAR POLYGONS AND CIRCLES



239. Regular polygons. — A *regular polygon* (See § 79) is equiangular and equilateral.

Regular polygons occur in nature, as in honeycomb. They are used in designs, as in tile floors, etc. The figure shows regular hexagons in a lace pattern.

240. Theorem. — *A circle may be circumscribed about any regular polygon.*



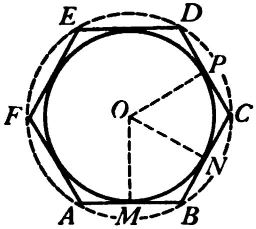
Hypothesis. $ABCD \cdots$ is any regular polygon.

Conclusion. A circle may be circumscribed about $ABCD \cdots$.

Proof. 1. $ABCD \cdots$ is a regular polygon. Hyp.

2. A circle may be drawn through A, B, and C. § 152
3. Let its center be O. Draw OA, OB, OC, OD.
4. $OB = OC$. § 151
5. $\therefore \angle 1 = \angle 2$. § 70
6. $\angle ABC = \angle BCD$ and $AB = CD$. Def. reg. poly.
7. $\therefore \angle 3 = \angle 4$. Ax. III
8. $\therefore \triangle AOB \cong \triangle COD$. § 63
9. $\therefore OA = OD$, and OD is a radius. Def. congruence
10. $\therefore$ the circle passes through D. Def. radius
11. Similarly, it goes through E, etc., and is circumscribed.

241. Theorem. — *A circle may be inscribed in any regular polygon.*



Hypothesis. $ABCD \cdots$ is any regular polygon.

Conclusion. A circle may be inscribed in $ABCD \cdots$.

Suggestions. A circle with center O may be circumscribed about $ABCD \cdots$. Why? Draw $OM \perp AB$, $ON \perp BC$, $OP \perp CD$, etc. Prove $OM = ON = OP =$ etc. Now draw a circle with center O and radius OM. Prove AB, BC, CD, etc., tangents to it.

242. Corollary 1. — *The inscribed and circumscribed circles of a regular polygon are concentric.*

This was shown in the proof of § 241.

243. Parts of a regular polygon. — The **center** of a regular polygon is the common center of the inscribed and circumscribed circles.

The **radius** of a regular polygon is the radius of the circumscribed circle.

The **apothem** of the polygon is the radius of the inscribed circle.

The **angle at the center** of a regular polygon is the angle between the radii drawn to the extremities of any side.

244. Corollary 2. — *The angle at the center of any regular polygon is equal to* 360° *divided by the number of sides.*

Why?

EXERCISES

1. How many degrees are there in each angle of a regular polygon of 10 sides? Of 16 sides? Of 20 sides?

2. How many degrees are there in the angle at the center of a regular pentagon? Of a regular hexagon? Of a regular octagon? Of a regular decagon?

3. What is the relation between the angle at the center of a regular polygon and the angle at a vertex? Give proof.

4. Prove that if all diagonals are drawn from any vertex of a regular polygon, they divide the angle at that vertex into equal parts.

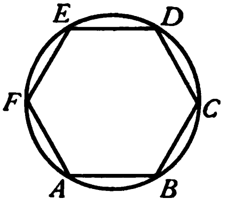
5. Prove that the radius drawn to any vertex of a regular polygon bisects the angle at that vertex.

6. Prove that the apothem of a regular polygon bisects any side to which it is drawn.

7. Prove that the apothem of an equilateral triangle is equal to one half of the radius.

8. Prove that the apothem of an equilateral triangle is equal to one third of the altitude of the triangle.

245. Theorem. — *If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon of that number of sides.*



Hypothesis. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} =$ etc.

Conclusion. $ABCD \cdots$ is a regular polygon, having as many sides as there are arcs AB, etc.

Suggestions. Why are the sides equal? Prove arc BDF = arc CEA etc. Then $\angle A = \angle B$, etc. Why? Write the proof.

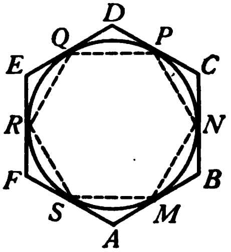
246. Corollary 1.—*An equilateral polygon inscribed in a circle is regular.*

The proof is left to the student.

247. Corollary 2.—*If from the vertices of a regular inscribed polygon line-segments are drawn to the middle points of the arcs subtended by the sides, they form another regular inscribed polygon of twice as many sides.*

The proof is left to the student.

248. Theorem.—*If a circle is divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.*

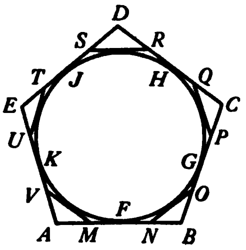


Hypothesis. $\widehat{MN} = \widehat{NP} = \widehat{PQ} =$ etc., and AB, BC, CD, etc., are tangents at M, N, P, etc., respectively.

Conclusion. $ABCD \cdots$ is a regular polygon, having as many sides as there are arcs MN, etc.

Suggestions. Draw MN, NP, etc. Now $\angle BMN = \angle MNB = \angle CNP =$ etc. Why? Then prove that $\triangle MBN$, $\triangle NCP$, etc., are congruent isosceles triangles, and hence that $MB = BN = NC = CP =$ etc. Then prove $AB = BC = CD =$ etc. Why does $\angle A = \angle B = \angle C =$ etc.?

Write the proof in full.



249. Corollary 1. — *Tangents drawn at the middle points of the arcs into which a circle is divided by the points of contact of the sides of a regular circumscribed polygon form with those sides a regular polygon of twice as many sides.*

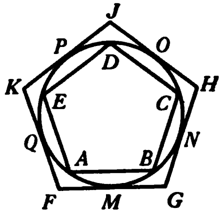
The proof is left to the student.

250. Corollary 2. — *If tangents are drawn at the vertices of a regular inscribed polygon, they form a regular circumscribed polygon of the same number of sides.*

The proof is left to the student.

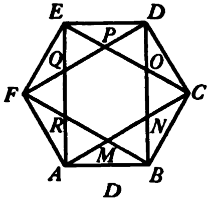
EXERCISES

1. The perimeter of a regular inscribed polygon is less than that of a regular inscribed polygon of twice as many sides.



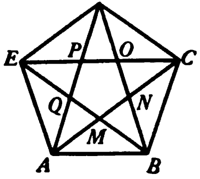
2. The perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of twice as many sides.

3. Tangents drawn at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.



4. The diagonals *AC*, *BD*, *CE*, etc., of a regular hexagon *ABCDEF* form another regular hexagon *MNOPQR*.

SUGGESTION. — Prove diagonals *AC*, *BD*, etc., equal. Then prove that a circle can be inscribed in hexagon *MNOPQR*.

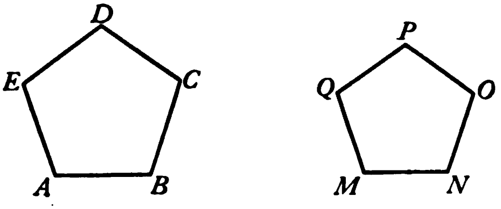


5. The diagonals of a regular pentagon *ABCDE* form another regular pentagon *MNOPQ*.

6. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

7. An equiangular polygon circumscribed about a circle is regular.

251. Theorem. — *Two regular polygons of the same number of sides are similar.*



Hypothesis. $ABCD \cdots$ and $MNOP \cdots$ are regular polygons of the same number of sides.

Conclusion. $ABCD \cdots \sim MNOP \cdots$.

Proof. 1. $ABCD \cdots$ and $MNOP \cdots$ are regular polygons of the same number of sides. Hyp.

2. $\therefore \angle A = \angle B =$ etc., and $\angle M = \angle N =$ etc. Def. reg. poly.

3. $\angle A + \angle B +$ etc. $= \angle M + \angle N +$ etc. § 101 and Ax. I

4. $\therefore$ if n is the number of sides of each polygon, $n \angle A = n \angle M$. Ax. XII

5. $\therefore \angle A = \angle M$. Ax. V

6. $\therefore \angle B = \angle N$, $\angle C = \angle O$, etc. Ax. I

7. $AB = BC =$ etc., and $MN = NO =$ etc. Def. reg. poly.

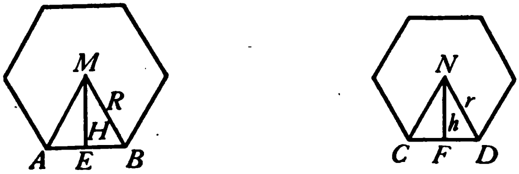
8. $\therefore \frac{AB}{MN} = \frac{BC}{NO} =$ etc. Ax. V

9. $\therefore ABCD \cdots \sim MNOP \cdots$. Def. sim. poly.

252. Corollary. — *The areas of two regular polygons of the same number of sides are to each other as the squares of their sides.*

The proof is left to the student.

253. Theorem. — *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Hypothesis. AB and CD are sides, and M and N the centers, respectively, of two regular polygons of the same number of sides. P and p are the perimeters, R and r the radii, and H and h the apothems, respectively, of the polygons.

Conclusion. $\frac{P}{p}=\frac{R}{r}=\frac{H}{h}$.

Suggestions. The polygons are similar. Why?

Hence $\frac{P}{p}=\frac{AB}{CD}$. Why?

By aid of § 244, prove $\triangle ABM \sim \triangle CDN$.

Then $\frac{AB}{CD}=\frac{R}{r}$. Explain.

Prove $\triangle EBM \sim \triangle FDN$.

Then $\frac{R}{r}=\frac{H}{h}$. Explain.

Hence $\frac{P}{p}=\frac{R}{r}=\frac{H}{h}$. Why?

Write out the proof.

254. Corollary. — *The areas of two regular polygons of the same number of sides are to each other as the squares of the radii, or of the apothems.*

The proof is left to the student.

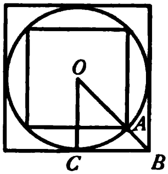
EXERCISES

1. Find the ratio of the perimeters and the ratio of the areas of two regular hexagons if their sides are 2 in. and 6 in., respectively.

2. Squares are inscribed in two circles of radii 2 in. and 8 in., respectively. Find the ratio of the perimeters of the squares and also of their areas.

3. What is the relation between the perimeters of the inscribed and circumscribed equilateral triangles of a circle?

4. What is the relation between the areas of the inscribed and circumscribed equilateral triangles of a circle?

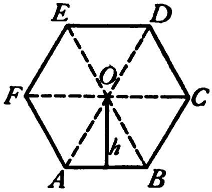


5. Prove that the area of an inscribed square equals one half of the area of a circumscribed square of a circle.

SUGGESTION. — Take the circumscribed square with its vertices on the radii produced of the inscribed square. Then $OC = CB$. Hence $\overline{OB}^2 = 2\,\overline{OC}^2 = 2\,\overline{OA}^2$.

6. Prove that the area of a regular inscribed hexagon equals three fourths of the area of a regular circumscribed hexagon of a circle.

255. Theorem. — *The area of any regular polygon equals one half of the product of its apothem and perimeter.*



Hypothesis. $ABCD \cdots$ is a regular polygon with apothem h and perimeter p.

Conclusion. Area of $ABCD \cdots = \frac{1}{2} hp$.

Proof. 1. $h =$ apothem and $p =$ perimeter of regular polygon $ABCD \cdots$. Hyp.

2. Draw radii OA, OB, OC, etc.

3. $\triangle OAB = \frac{1}{2} h \times AB$, $\triangle OBC = \frac{1}{2} h \times BC$, etc. § 219

4. $\therefore \triangle OAB + \triangle OBC + \text{etc.} = \frac{1}{2}h \times AB + \frac{1}{2}h \times BC + \text{etc.}$ Ax. II

5. $\therefore \triangle OAB + \triangle OBC + \text{etc.} = \frac{1}{2}h(AB + BC + \text{etc.})$. Factoring

6. $\therefore$ Area of $ABCD \cdots = \frac{1}{2}hp$. Ax. XII

EXERCISES

If R is the radius of a circle:

1. The side of an inscribed square equals $R\sqrt{2}$.

2. The area of an inscribed square equals $2R^2$.

3. The side of an inscribed equilateral triangle equals $R\sqrt{3}$.

4. The area of an inscribed equilateral triangle equals $\frac{3}{4}R^2\sqrt{3}$.

5. The side of a circumscribed equilateral triangle equals $2R\sqrt{3}$.

6. The area of a circumscribed equilateral triangle equals $3R^2\sqrt{3}$.

7. The side of a regular inscribed hexagon is R.

8. The area of a regular inscribed hexagon equals $\frac{3}{2}R^2\sqrt{3}$.

9. The side of a regular circumscribed hexagon equals $\frac{2}{3}R\sqrt{3}$.

10. The area of a regular circumscribed hexagon equals $2R^2\sqrt{3}$.

11. If the area of a square is 400 sq. in., find the apothem and radius.

12. If the area of an equilateral triangle is $12\sqrt{3}$, find its radius.

SUGGESTION. — $\frac{3}{4}R^2\sqrt{3} = 12\sqrt{3}$.

13. If the area of an equilateral triangle is $108\sqrt{3}$, find its apothem.

If the radius of a circle is 24 in., find:

14. The area of an inscribed equilateral triangle.

15. The area of a circumscribed equilateral triangle.

16. The area of an inscribed square.

17. The area of a circumscribed square.

18. The area of an inscribed regular hexagon.

19. The area of a circumscribed regular hexagon.

20. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

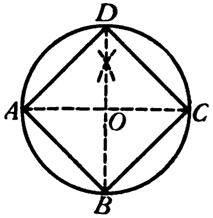
21. If the diagonals joining the alternate vertices of a regular hexagon are drawn, the area of the second regular hexagon which they form is one third that of the original hexagon.

22. If from any point within a regular polygon of n sides perpendiculars are drawn to all of the sides, the sum of these perpendiculars is equal to n times the apothem of the polygon.

SUGGESTION. — Join the point to each vertex. Then get two expressions for the area of the polygon, and put them equal to each other.

23. The stock from which tools are made is often in the form of round or cylindrical rods. From a round rod a tool is to be cut so that one end is a regular hexagon whose side is $\frac{7}{8}$ in. What size (diameter) of stock must be selected from which to cut it?

256. Construction. — *Inscribe a square in a given circle.*



Given a circle with center O.

Required to inscribe a square in the circle.

Construction. 1. Draw two perpendicular diameters, AC and BD. Draw AB, BC, CD, and DA.

2. Then $ABCD$ is the required square.

Proof. The proof is left to the student.

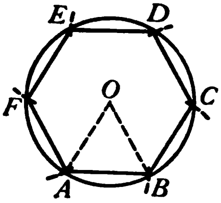
257. Corollary 1. — *Regular polygons of* 4, 8, 16, 32, *etc., sides may be inscribed in a circle.*

How? Use § 247.

258. Corollary 2. — *By drawing tangents at the vertices of regular inscribed polygons of* 4, 8, 16, 32, *etc., sides, regular polygons of the same numbers of sides may be circumscribed about a circle.*

This follows from § 250 and § 257.

259. Construction. — *Inscribe a regular hexagon in a given circle.*



Given a circle with center O.

Required to inscribe a regular hexagon in the circle.

Construction. 1. Mark any point A on the circle. With center at A and radius equal to that of the circle, draw an arc intersecting the circle at B.

2. Draw AB.

3. Then AB is a side of the required hexagon, and by applying it six times as a chord, the regular hexagon $ABCDEF$ may be inscribed.

Suggestions. $\triangle OAB$ is equilateral. Hence $\angle AOB = 60°$, and is contained exactly six times about O. Why? Prove that $\overset{\frown}{AB}$ is contained exactly six times in the circle. Apply § 245.

Make the construction, and write out the construction and proof.

260. Corollary 1. — *By joining the alternate vertices of a regular inscribed hexagon, an equilateral triangle may be inscribed in a circle.*

The proof is left to the student. Make the construction.

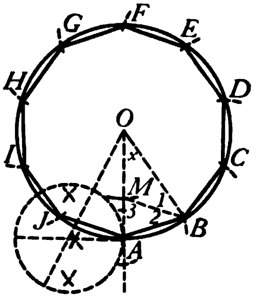
261. Corollary 2. — *Regular polygons of* 3, 6, 12, 24, *etc., sides may be inscribed in a circle.*

How? Use § 247.

262. Corollary 3. — *By drawing tangents at the vertices of regular inscribed polygons of* 3, 6, 12, 24, *etc., sides, regular polygons of the same numbers of sides may be circumscribed about a circle.*

This follows from § 250 and § 261. Explain.

263. Construction. — *Inscribe a regular decagon in a given circle.*



Given a circle with center O.

Required to inscribe a regular decagon in the circle.

Construction. 1. Draw any radius OA, and divide it in extreme and mean ratio at M.

2. Beginning at A, mark off ten arcs AB, BC, etc., with chords each equal to OM, the longer segment of OA. Draw the chords of these arcs.

3. These chords form the required regular decagon $ABCD \cdots$.

Proof. 1. Draw OB and MB.

2. $\frac{OA}{OM} = \frac{OM}{MA}$, and $OM = AB$. Construction

3. $\therefore \frac{OA}{AB} = \frac{AB}{MA}$. Ax. XII

4. $\therefore \triangle MAB \sim \triangle ABO$. § 130

5. $\therefore \angle 3 = \angle OBA$ and $\angle 2 = \angle x$. Def. sim. poly.

6. But $OB = OA$. § 151

7. $\therefore \angle BAO = \angle OBA$. § 70

8. $\therefore \angle 3 = \angle BAO$. Ax. I

9. $\therefore MB = AB$. § 74

10. $\therefore MB = OM$. Ax. I

11. $\therefore \angle 1 = \angle x$. § 70

12. $\angle OBA = 2 \angle x$ and $\angle BAO = 2 \angle x$. Ax. XII

13. $\therefore$ in $\triangle ABO$, $5 \angle x = 180°$. § 48

14. $\therefore \angle x = 36°$. Ax. V

15. $\therefore \overset{\frown}{AB} = 36°$, or exactly one tenth of the circle. § 173

16. Since chords AB, BC, CD, etc., were constructed equal, $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} =$ etc. § 151, (7)

17. $\therefore$ each arc, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, etc., equals one tenth of the circle, or the ten arcs constitute the circle. Ax. I

18. $\therefore ABCD \cdots$ is a regular decagon. § 245

264. Corollary 1. — *By joining the alternate vertices of a regular inscribed decagon, a regular pentagon may be inscribed in a circle.*

Make the construction and give proof.

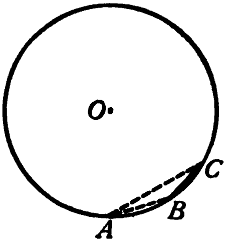
265. Corollary 2. — *Regular polygons of* 5, 10, 20, 40, *etc., sides may be inscribed in a circle.*

How? Give proof.

266. Corollary 3. — *By drawing tangents at the vertices of regular inscribed polygons of* 5, 10, 20, 40, *etc., sides, regular polygons of the same numbers of sides may be circumscribed about a circle.*

Give proof.

267. Construction. — *Inscribe a regular polygon of* 15 *sides in a given circle.*



Given a circle with center O.

Required to inscribe in the circle a regular polygon of 15 sides.

Construction. 1. Draw AB, the side of a regular inscribed decagon, and AC, the side of a regular inscribed hexagon, and draw BC.

2. Then BC is the side of the required polygon.

Suggestion. $\overset{\frown}{AC}$ is what part of the circle? $\overset{\frown}{AB}$ is what part of the circle? Then what part of the circle is $\overset{\frown}{BC}$?

268. Corollary 1. — *Regular polygons of* 15, 30, 60, *etc., sides may be inscribed in a circle.*

How? Give proof.

269. Corollary 2. — *By drawing tangents at the vertices of regular inscribed polygons of* 15, 30, 60, *etc., sides, regular polygons of the same numbers of sides may be circumscribed about a circle.*

The proof is left to the student.

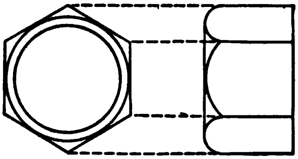
270. Summary. — The methods of constructing regular polygons of 3, 6, 12, etc., or 4, 8, 16, etc., or 5, 10, 20, etc., or 15, 30, 60, etc., sides by use of circles have been shown in the preceding sections. Until the beginning of the last century it was thought that these were the only regular polygons possible of construction by means of a *straightedge* and *compasses.* But a German mathematician named Gauss published a proof in 1801 that a regular polygon of 17 sides and certain other regular polygons could be so constructed.

EXERCISES

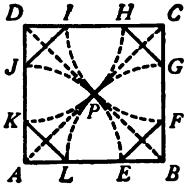
1. Construct regular polygons of 5 sides, 6 sides, 8 sides, 10 sides, and 12 sides.

2. In making a water wheel, a square block of wood is to be made into the form of a regular octagon by cutting off the four corners, and then attaching buckets to each of the eight faces. Show how to cut off the corners accurately.

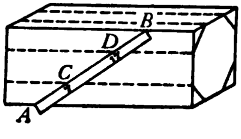
3. Draw the face and edge views of a hexagonal nut of a bolt, the side of the nut to be one inch.



4. The following method of obtaining a regular octagon from a square is much used in practical work: Draw the diagonals of the square, intersecting at P. With radius equal to AP and centers A, B, C, D, draw arcs cutting the sides of the square at E, F, G, H, I, J, K, L. Draw EF, GH, IJ, and KL. Then $EFGHIJKL$ is a regular octagon. Prove it.

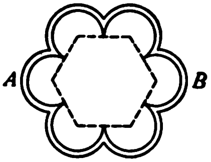


5. Carpenters, in order to cut a square piece of timber down to an octagonal shape, proceed as follows: Place a 24-inch rule diagonally across the timber, the ends even with the edges at A and B, as in the figure. Then mark the points C and D at the 7-inch and 17-inch points on the rule. Through these points draw lines parallel to the edges of the timber. Repeat this on each face. The corners must be cut down to these lines to form an octagonal piece of timber. Is this method accurate or approximate? Prove the answer.

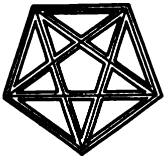


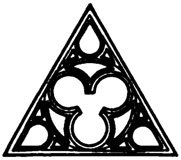
6. The top of a taboret is to be made in the form of a regular octagon whose longest diagonal is 12 in. Construct a design of it to the scale of 3 in. to an inch.

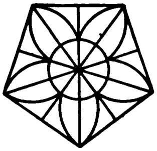
7. Construct a paper pattern for a doily like the adjoining figure, making the width from A to B 12 in.

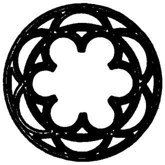


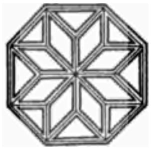
8–14. Study the following ornamental designs, and explain how they are made. Construct such designs similar to these as the teacher may direct, making the drawings several times as large as these.

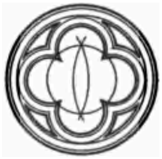


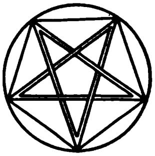






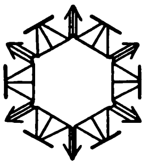


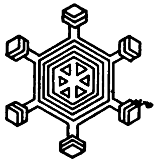


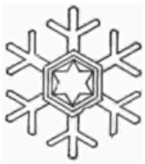


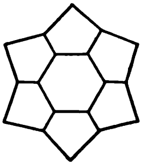
15. Observe the patterns of linoleums, metal ceiling panels, tile floors, etc., ornamental windows, and furniture designs. Make large drawings of one or more of the designs observed.

16. The following designs show different forms of snow crystals. Explain how they may be constructed, and make on a large scale a drawing of one of them.









17. If the radius of a regular decagon is R, prove that its side is equal to $\frac{1}{2}R(\sqrt{5} - 1)$.

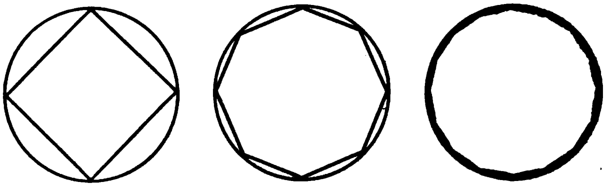
Suggestion. — Let s represent the side. Then, by the construction of extreme and mean ratio in § 263, $\frac{R}{s} = \frac{s}{R - s}$.

18. If the radius of a regular decagon is R, prove that its apothem is equal to $\frac{1}{4}R\sqrt{10 + 2\sqrt{5}}$.

19. From the expressions for the side and apothem in Ex. 17 and Ex. 18, find the formula for the area of a regular decagon whose radius is R.

MEASUREMENT OF THE CIRCLE

271. Measurement of the circle. — The *length* of a circle cannot be measured like that of a line-segment, because a linear unit of measure cannot be laid off along a circle. Nevertheless, a circle has a definite length, which can be measured *approximately* in terms of a linear unit.



Thus, if regular polygons of 4, 8, 16, etc., sides are inscribed in a circle, it is evident that the perimeter of each is less than the circle; but the greater the number of sides the polygon has, the nearer its perimeter approaches to the circle. Hence it is assumed that a circle has a *definite length*, which can be approximated as nearly as we please by taking a perimeter of a regular inscribed polygon with a sufficiently great number of sides.

The *length* of a circle is called the **circumference** of the circle.

The area of the surface inclosed by a circle is called the **area of the circle**.

The **area** of a circle, like the circumference, can be measured only approximately.

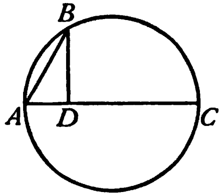
The measurement of the circle requires the use of certain principles about *variables*, *constants*, and *limits*, which are discussed in the following sections.

272. Variables and constants. — A **variable quantity**, or simply a **variable**, is a quantity whose value is continually changing.

A **constant quantity**, or simply a **constant**, is a quantity whose value remains unchanged.

Thus, one's age is a *variable*, while the number of days in a week is *constant*.

Show that as the chord AB revolves about the fixed point A, its projection AD on the diameter AC *varies* in length from zero to the length of the diameter. Show that at the same time the distance of the chord from the center varies from the length of the radius to zero. What line-segment in the figure is constant?



Suggest other examples of variables and constants.

273. Limit of a variable. — Quantities may vary in different ways. Some variables are said to change in such ways as to *approach constants as limits.*

The **limit** of a variable is a constant which the variable continually approaches and from which it may be made to differ by less than any fixed positive quantity however small.

Thus, suppose a point P on AB to move from A toward B, under the condition that during the first second it should move one half of the distance, during the next second one half of the remaining distance, and so on. It is evident that by continuing long enough P can be made to

approach as near B as we please, although it will never reach B. In that case, AP is a variable which approaches indefinitely near to the constant distance AB, or approaches AB as a *limit.*

Again, consider the series of fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, etc. If x is a literal number which assumes in turn the value of each of these fractions, beginning with the smallest, it is evident that x is a variable which increases toward 1 and is made to differ from 1 by less than any assumed value however small. Hence x approaches 1 as a limit.

The expression "x approaches a as a limit" is written $x \doteq a$.

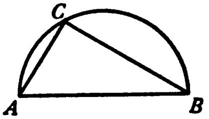
EXERCISES

1. In the figure of § 273, is PB a variable or constant? What is its limit?

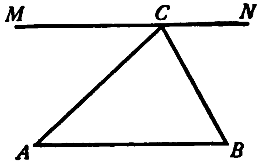
2. Show that the difference between 1 and x in the example of § 273 can be made less than $\frac{1}{100}$; less than $\frac{1}{1000}$; less than one millionth. What is the limit of $1 - x$?

3. If x increases indefinitely, what is the limit of $\frac{1}{x}$?

4. Angle ACB is inscribed in a semicircle with diameter AB. As point C moves along the arc from A toward B, is $\angle ACB$ constant or variable? Why?



5. The base AB of $\triangle ABC$ is fixed, but the vertex C moves on line MN, which is parallel to AB. Is $\angle ACB$ constant or variable? For what position of C does it have the greatest value? What is the limit approached by $\angle ACB$ as C moves off indefinitely in the direction CN?



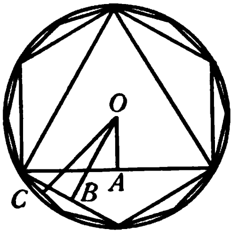
6. If the number of sides of a regular circumscribed polygon is increased indefinitely, does the perimeter approach a limit or not? What? Does the radius of the polygon approach a limit? What? What line-segment in the polygon is constant? (Proofs of the answers are not expected.)

7. If a variable arc approaches zero as a limit, the subtended chord approaches what limit?

8. If the altitude of an isosceles triangle having a fixed base is increased indefinitely, what limit does the angle at the vertex approach? What limit does each of the angles at the base approach? What limits do the respective angles approach as the altitude gets shorter and shorter and approaches zero?

9. Zeno, a member of one of the ancient Greek schools, proposed the following: He reasoned that "if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 yards start, it could never be overtaken; for, when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on forever: thus Achilles would get nearer and nearer to the tortoise but never overtake it." Was Zeno's reasoning correct? If not, what was the fallacy in his reasoning?

274. Limits applied to regular polygons. — Inscribe in a circle a regular polygon. By joining its vertices to the middle points of the arcs subtended by the sides, inscribe a regular polygon of twice as many sides. Obtain in a similar manner a third regular polygon of twice as many sides as the second, and so on. Let their apothems be OA, OB, OC, etc. Then $OB > OA$, $OC > OB$, etc. It is evident that:

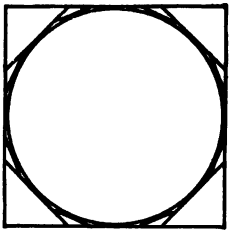


(1) *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as a limit.*

(2) *If the number of sides of a regular inscribed polygon is indefinitely increased, the perimeter approaches the circumference of the circle as a limit.*

Circumscribe about a circle a regular polygon. By use of § 249, circumscribe a regular polygon of twice as many sides, then one of four times as many sides, etc. From this figure it is evident that:

(3) *If the number of sides of a regular circumscribed polygon is indefinitely increased, the perimeter of the polygon approaches the circumference of the circle as a limit.*

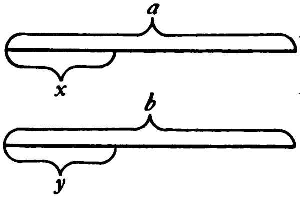


(4) *If the number of sides of a regular circumscribed polygon is indefinitely increased, the area of the polygon approaches the area of the circle as a limit.*

275. Principles of limits. — The following general principles regarding variables and their limits are assumed as evident:

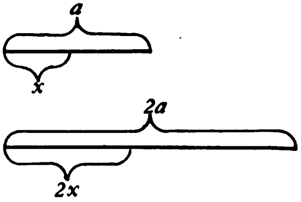
(1) *If two variables are equal and each approaches a limit, the limits are equal.*

Thus, if two variables x and y are always equal, and $x \doteq a$ while $y \doteq b$, then $a = b$.



(2) *If the limit of a variable* ***x*** *is* ***a****, then the limit of* ***kx*** *is* ***ka****, where* ***k*** *is any constant.*

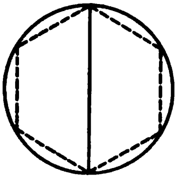
For example, if $x \doteq a$, that is, if x approaches a so that the difference between a and x becomes and remains less than any positive quantity however small, then at the same time $2x$ must approach $2a$ so that the difference between $2a$ and $2x$ becomes and remains less than any positive quantity however small, that is, $2x \doteq 2a$. Similarly, $3x \doteq 3a$, $4x \doteq 4a$, etc.

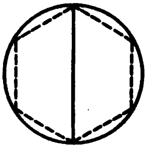


(3) *If the limit of a variable* ***x*** *is* ***a****, then the limit of* $\frac{x}{k}$ *is* $\frac{a}{k}$*, where* ***k*** *is any constant.*

This follows from (2), because $\frac{x}{k}$ may be written $\frac{1}{k} \times x$, or a constant times x.

276. Theorem. — *The ratio of the circumference to the diameter of a circle is constant, or the same for any two circles.*





Hypothesis. C and c are the circumferences of two circles whose diameters are D and d, respectively.

Conclusion. $\frac{C}{D} = \frac{c}{d}$.

Proof. 1. C and c, and D and d, are the circumferences and diameters respectively of two circles. Hyp.

2. Inscribe in the circles regular polygons of the same number of sides. Let P and p be the perimeters and R and r the radii of the polygons in circles with diameters D and d, respectively.

3. Then $\frac{P}{p} = \frac{R}{r}$. § 253

4. But $\frac{R}{r} = \frac{2R}{2r} = \frac{D}{d}$. Reduct. and Ax. XII

5. $\therefore \frac{P}{p} = \frac{D}{d}$. Ax. I

6. $\therefore \frac{P}{D} = \frac{p}{d}$. § 118, (4)

7. Now if the number of sides of the polygons is indefinitely increased, $P \doteq C$ and $p \doteq c$. § 274, (2)

8. $\therefore \frac{P}{D} \doteq \frac{C}{D}$ and $\frac{p}{d} \doteq \frac{c}{d}$. § 275, (3)

9. $\therefore \frac{C}{D} = \frac{c}{d}$. § 275, (1)

277. The Greek letter π. — The constant ratio of the circumference to the diameter of a circle is denoted by the Greek letter π (pī). If C is the circumference and D the diameter, $\frac{C}{D} = \pi$.

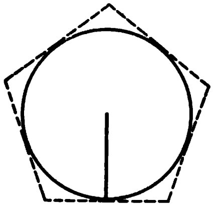
278. Corollary 1. — *In any circle,* $C = \pi D = 2\pi R$.

This follows directly from § 277.

279. Corollary 2. — *The circumferences of any two circles are to each other as their diameters or as their radii.*

For $\frac{C}{c} = \frac{\pi D}{\pi d} = \frac{D}{d} = \frac{R}{r}$.

280. Theorem. — *The area of a circle is equal to one half of the product of its radius and circumference.*



Hypothesis. A is the area of a circle whose circumference is C and radius is R.

Conclusion. $A = \frac{1}{2}RC$.

Proof. 1. A = area, C = circumference, R = radius. Hyp.

2. Circumscribe a regular polygon about the circle. Let S = area and P = perimeter of the polygon.

3. Then $S = \frac{1}{2}RP$. § 255

4. Now if the number of sides of the polygon is indefinitely increased, $S \doteq A$ and $P \doteq C$. § 274, (4), (3)

5. $\therefore \frac{1}{2}RP \doteq \frac{1}{2}RC$. § 275, (2)

6. $\therefore A = \frac{1}{2}RC$. § 275, (1)

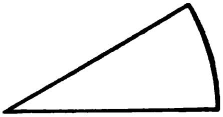
281. Corollary 1. — *The area of a circle equals* πR^2.

For $A = \frac{1}{2} RC = \frac{1}{2} R \times 2\pi R = \pi R^2$.

282. Corollary 2. —*The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

For $\frac{A}{a} = \frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2} = \frac{D^2}{d^2}$.

283. A sector of a circle. — The figure formed by an arc of a circle and the two radii drawn to its extremities is called a **sector of a circle.**



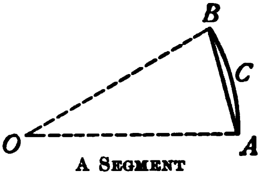
A SECTOR

284. Corollary 3. — *The area of a sector of a circle is equal to one half of the product of its radius and its arc.*

The proof is similar to that of § 280.

285. A segment of a circle. — The figure formed by a chord of a circle and its subtended arc is called a **segment of a circle.**

The area of a segment ACB may be obtained by subtracting the area of $\triangle OAB$ from the area of sector $OACB$.



A SEGMENT

EXERCISES

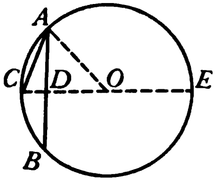
1. Show that if the diameter of a circle is D, the area equals $\frac{1}{4}\pi D^2$.

2. Prove that if the radius of a circle is R, the area of a segment of the circle whose arc is a quadrant is equal to $\frac{1}{4} R^2(\pi - 2)$.

3. Prove that if the radius of a circle is R, the area of a segment of the circle whose arc is 60° is equal to $\frac{1}{12} R^2(2\pi - 3\sqrt{3})$.

4. Prove that the area of a segment of a circle is equal to the area of the sector having the same arc, minus one half of the product of the square of the radius and the sine of the central angle subtended by the arc.

286. Theorem. — *If r is the radius of a circle and s the side of a regular inscribed polygon, then the side of a regular inscribed polygon of twice as many sides is* $\sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.



Hypothesis. The radius of a circle is r, and the side of a regular inscribed polygon is s.

Conclusion. The side of a regular inscribed polygon of twice as many sides is $\sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.

Suggestions. Let $AB = s$, and let AC equal the side of a regular polygon of twice as many sides. Draw radius AO and draw diameter CE intersecting AB at D.

Prove that CE is the perpendicular bisector of AB.

In $\triangle ADO$, show that $\overline{DO}^2 = r^2 - \frac{1}{4}s^2$.

Hence $DO = \sqrt{r^2 - \frac{1}{4}s^2}$, and $CD = r - \sqrt{r^2 - \frac{1}{4}s^2}$. Why?

Now by § 193, $\overline{AC}^2 = CE \times CD = 2r(r - \sqrt{r^2 - \frac{1}{4}s^2})$. Explain.

Hence, $AC = \sqrt{2r(r - \sqrt{r^2 - \frac{1}{4}s^2})} = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.

Write the proof in full.

287. Computation of π. — By aid of the formula in § 286, the approximate value of π may now be computed.

For this purpose, in a circle of radius $r = 1$, let regular polygons of 6, 12, 24, 48, etc., sides be inscribed.

Let s_6 = side and p_6 = perimeter of the polygon of 6 sides,
s_{12} = side and p_{12} = perimeter of the polygon of 12 sides, etc.

Then $s_6 = 1$ and $p_6 = 6$. Why?

Hence, by the formula in § 286, the following values may be computed in succession:

$$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}} = .51763809. \quad \therefore p_{12} = 6.21165708.$$

$$s_{24} = \sqrt{2 - \sqrt{4 - .51763809^2}} = .26105238. \quad \therefore p_{24} = 6.26525722.$$

$$s_{48} = \sqrt{2 - \sqrt{4 - .26105238^2}} = .13080626. \quad \therefore p_{48} = 6.27870041.$$

$$s_{96} = \sqrt{2 - \sqrt{4 - .13080626^2}} = .06543817. \quad \therefore p_{96} = 6.28206396.$$

$$s_{192} = \sqrt{2 - \sqrt{4 - .06543817^2}} = .03272346. \quad \therefore p_{192} = 6.28290510.$$

$$s_{384} = \sqrt{2 - \sqrt{4 - .03272346^2}} = .01636228. \quad \therefore p_{384} = 6.28311544.$$

$$s_{768} = \sqrt{2 - \sqrt{4 - .01636228^2}} = .00818121. \quad \therefore p_{768} = 6.28316941.$$

By continuing this computation, greater and greater perimeters are found, and hence closer and closer approximations to the circumference. Taking 6.28316941, the perimeter of a polygon of 768 sides, as approximately equal to the circumference, and dividing it by 2, the diameter, gives 3.14159, nearly, the approximate value of π. The values of π used in computation in practical work are 3.1416, 3.142, 3.14, or $\frac{22}{7}$, depending upon the accuracy of the result required.

NOTE. — The exact value of π cannot be computed. Its value computed to 25 decimal places is 3.1415926535897932384626433. The correct approximate value has now been computed to over 700 decimal places.

The ratio π has a long and interesting history. The ancient Babylonians and Hebrews thought that $\pi = 3$, as shown by the Bible. (See 1 Kings vii, 23 and 2 Chronicles iv, 2.) Ahmes, an Egyptian, about 1700 B.C., used a value of π equivalent in modern notation to 3.1604. Archimedes, a Greek born 287 B.C., by employing inscribed and circumscribed regular polygons, proved the value of π to be between $3\frac{1}{7}$ and $3\frac{10}{71}$. Aryabhatta, a Hindu born 476 A.D., found $\pi = 3.1416$. Many other values of π have been used by different people in the past.

The attempt to compute the exact value of π was connected with the famous impossible problem with which mathematicians struggled for centuries, dating from the time of the ancient Greeks, namely, to construct with straightedge and compasses alone a square whose area should equal that of a given circle, or to "square a circle."

EXERCISES

1. The radius of a circle is 4 in. Find the circumference; the area.

2. The circumference of a circle is 24 ft. Find its diameter; its radius.

3. The area of a circle is 256 sq. in. Find its radius; its diameter; its circumference.

4. In a circle whose radius is 8 ft., find the length of an arc whose central angle is 36°.

5. Find the area of a sector of a circle whose central angle is 40° and radius 12 in.

6. Find the area of a segment of a circle whose arc is 60°, the radius being 3 ft.

7. A horizontal oil tank 8 ft. in diameter is filled with oil to a depth of 2 ft. Find the area of a segment formed by a vertical cross section of the oil.

8. How long is the tire of a carriage wheel that is 4 ft. in diameter?

9. The central angle whose arc is equal to the radius is often used as the unit of measure of angles. It is called a *radian*. Find the number of degrees in a radian.

10. In the papyrus written by Ahmes, the Egyptian, the area of a circle was found by subtracting from the diameter one ninth of its length and squaring the remainder. This was equivalent to using what value of π?

11. As stated in the note above, Archimedes proved that the value of π was between $3\frac{1}{7}$ and $3\frac{10}{71}$. Express these numbers decimally. They give the correct value of π to how many decimal places?

12. Show that the area of a circle is equal to that of a triangle whose base equals the circumference and altitude equals the radius.

13. How many revolutions per mile does a 28-inch bicycle wheel make?

14. A water tank for supplying locomotives is cylindrical in form and 20 ft. in diameter. How long must a piece of strap iron be cut to make a band around it, allowing 1 ft. for overlapping?

15. The earth is nearly 8000 miles in diameter. What is the approximate length of the equator?

16. How much belting does it require to run over two pulleys each 24 in. in diameter and with their centers 16 ft. apart?

17. A grindstone of Ohio stone will stand a surface speed of 2500 ft. per minute. How many revolutions per minute will a stone stand if it is 4 ft. in diameter?

18. An emery stone will safely stand a surface speed of 5500 ft. per minute. An emery grinder is to make 1500 revolutions per minute. What is the diameter of the largest wheel that may safely be used?

19. The driving pulley of an engine is 6 ft. in diameter and makes 120 revolutions per minute. It is belted to a 24-inch pulley on the main shaft that runs the machinery of a mill. Find the speed of the shaft (revolutions per minute).

20. A running track consists of two parallel straight stretches each a quarter of a mile long, and two semicircular ends each a quarter of a mile long at the inner curb. If two athletes run, one 5 ft. from the inner curb and the other 10 ft. from it, by how much is the second man handicapped?

21. If it is customary in iron turning to allow a cutting speed at the rim of 40 ft. per minute, at what speed (revolutions per minute) should a lathe be driven for turning a piece of iron 2 in. in diameter?

22. Through what angle must a 20-ft. rail of a railroad track be bent to fit a curve with a radius of 400 ft.?

23. How many tulips are required for a circular flower bed 8 ft. in diameter, if 36 sq. in. are allowed for each plant?

24. A cow is tied at the end of a rope 100 ft. long, which is attached to the corner of a barn 50 ft. square. Over how much area may the cow graze?

25. The boiler of an engine has 300 tubes, each 3 in. in diameter, for conducting the heat through the water. Find their total cross-sectional area.

26. The steam pressure of an engine is indicated as 96 lb. per square inch. The cylinder of the engine is 20 in. in diameter. What is the total pressure against the piston?

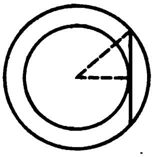
27. What is the propelling pressure exerted against the piston of a locomotive when the steam pressure is 100 lb. per square inch and the diameter of the cylinder is 28 in.?

28. Four drain pipes in a house, each 4 in. in diameter, empty into one large pipe. If the larger pipe is to have the same capacity as the four smaller pipes combined, what must be its diameter?

29. A water main in the street supplies 6 branch pipes, each 6 in. in diameter, and 4 branch pipes, each 8 in. in diameter. In order to have the same capacity as all of the branch pipes combined, what should be the diameter of the water main?

30. In a hot-air furnace for a dwelling, one pipe conducts cold fresh air to the furnace, and other pipes then conduct the warm air from the furnace to the different rooms. If there are 4 pipes, each 10 in. in diameter, and 2 pipes, each 12 in. in diameter, that conduct the warm air, what should be the diameter of the cold-air duct in order to have the same capacity?

31. The boiler of an engine has 96 flues, each 3 in. in diameter, which conduct the hot air from the furnace through the water to heat it. If the rule requires that the smokestack shall have the same capacity (cross-sectional area) as all of the flues, which empty the smoke into it, what must be the diameter of the smokestack?

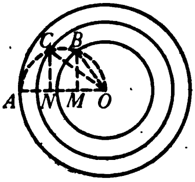


32. Prove that the area of the ring between two concentric circles whose radii are R and r respectively is $\pi(R + r)(R - r)$.

33. Prove that the area of the ring between two concentric circles is equal to the area of a circle whose diameter is a chord of the larger circle which is tangent to the smaller.

SUGGESTION.—In the formula of Ex. 32, the factors $(R + r)(R - r)$ equal the square of what single line-segment?

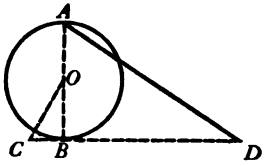
34. A convenient formula used in practical work for finding the area of a "hollow circle" or ring, is Area $= \pi \times \frac{D + d}{2} \times t$, where D is the outer diameter, d the inner diameter, and t the thickness of the ring (difference of the outer and inner radii). Prove this formula.



35. Divide a given circle into three equal parts by drawing two concentric circles.

SUGGESTION.—Trisect the radius at M and N. Draw a semicircle on AO as diameter. Erect perpendiculars MB and NC, meeting the semicircle at B and C. With centers at O and radii OB and OC, draw circles.

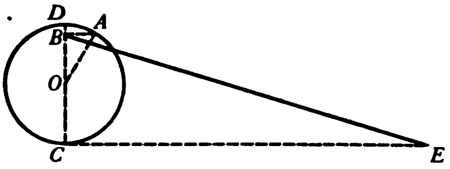
36. Mathematicians long struggled in the attempt to construct a line-segment equal to the circumference of a given circle. The following is Ceradini's *approximate* construction:



Draw diameter AB, and draw CD tangent at B, making $\angle COB = 30°$ and $CD = 3$ times the radius. Draw AD. Then $2\,AD$ = the circumference, nearly. Determine the accuracy of this construction by computing the ratio of $2\,AD$ to AB.

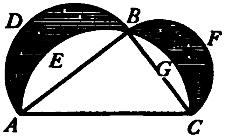
SUGGESTION. — Let R = radius. Compute AB and $2\,AD$ in terms of R, then divide.

37. A line-segment approximately equal to the circumference of a given circle may be constructed as follows:

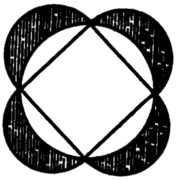


Draw diameter CD. Construct $\angle AOD = 30°$. Draw $AB \perp CD$. Draw CE tangent at C and equal to $3\,CD$. Draw BE. Then BE = the circumference, approximately. Determine the accuracy of this construction.

38. Semicircles are constructed on the three sides of a right triangle, as in the figure. Show that the sum of the *crescents* $ADBE$ and $BFCG$ is equal to the triangle.



39. To Hippocrates, a Greek born 430 B.C., is due the following: If on the sides of an inscribed square as diameters semicircles are described, the area of the four crescents lying without the circle equals the area of the inscribed square. Give the proof.



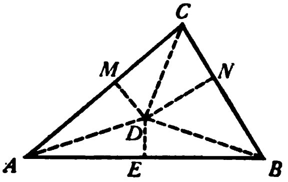
40. If the diameters of the circles whose arcs form the trefoil in Ex. 11, page 186, are each 10 in., find the perimeter of the trefoil.

41. Find the area of the trefoil in Ex. 40, that is, the area inclosed by the three arcs.

MISCELLANEOUS EXERCISES FOR REVIEW

NOTE. — A list of exercises for review of the first seven chapters is given at the end of Chapter VII. Of the following exercises, the first thirteen are based also on the first seven chapters; the remainder are based on Chapters VIII, IX, X, and XI.

1. Find the fallacy in the following proof that every triangle is isosceles:



HYPOTHESIS. — $\triangle ABC$ is any triangle.

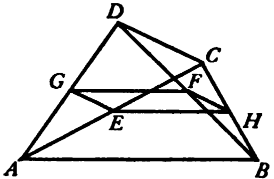
CONCLUSION. — $\triangle ABC$ is isosceles.

PROOF. — 1. Draw the bisector CD of $\angle C$ and the perpendicular bisector DE of AB, and let them intersect at D. Draw $DM \perp AC$ and $DN \perp BC$.

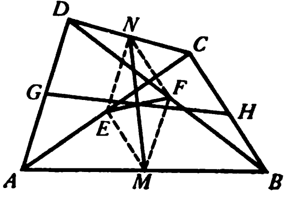
2. Then $DA = DB$. §105
3. And $DM = DN$. §108
4. $\therefore \triangle ADM \cong \triangle DBN$. §77
5. $\therefore AM = BN$. Def. congruence
6. Also $\triangle MDC \cong \triangle DNC$. §77
7. $\therefore MC = NC$. Def. congruence
8. $\therefore AM + MC = BN + NC$ or $AC = BC$. Ax. II
9. $\therefore \triangle ABC$ is isosceles. Def. isos. △

2. Construct a parallel to the side AB of $\triangle ABC$, meeting AC at D and BC at E, so that $DE = BE$.

3. The line which joins the middle points of two sides of a triangle bisects any line-segment drawn to the third side from the opposite vertex.



4. If E and F are the middle points of diagonals AC and BD, respectively, of any quadrilateral $ABCD$, and G and H the middle points of AD and BC, respectively, then $EHFG$ is a parallelogram.

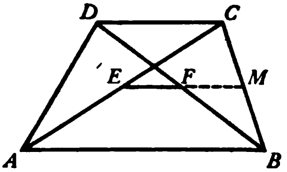


5. The lines joining the middle points of the opposite sides of any quadrilateral and the line joining the middle points of the diagonals of the quadrilateral meet at one point.

SUGGESTION. — Apply Ex. 4.

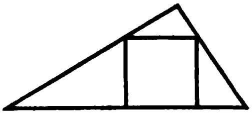
6. The line-segment joining the middle points of the diagonals of a trapezoid is parallel to the bases and equal to one half of their difference.

Suggestion. — Join E and F to M, the middle point of BC.



7. Find a formula for the side of a square inscribed in a triangle with altitude a and base b.

Suggestion. — Let x = side of square. Show that $\frac{a-x}{a} = \frac{x}{b}$. Solve for x.

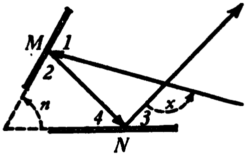


8. What is the side of a square inscribed in a triangle with altitude 12 in. and base 8 in.?

9. A median of a triangle is less than one half of the perimeter.

10. The following principle is used in optical instruments, such as the *sextant:* Two mirrors, M and N, are placed at an angle n with each other. A ray of light is reflected first by M, then by N, so that it then crosses its first path at an angle x. Prove that $\angle x = 2\angle n$.

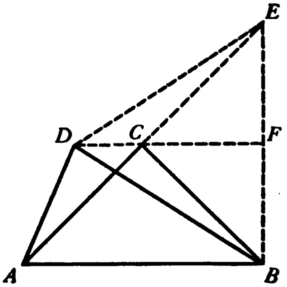
Suggestion. — By a law of light, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.



11. If the perpendicular bisectors of the sides of a triangle intersect within the triangle, the triangle is acute.

12. Of all triangles having the same base and equal altitudes, the isosceles triangle has the least perimeter.

Suggestion. — $AD + DB = AD + DE$ and $AC + CB = AC + CE$.



13. Of all triangles of equal areas, the equilateral triangle has the least perimeter.

14. The tangent to a circle at the middle point of an arc is parallel to the chord which subtends the arc.

15. If $ABCD$ is a quadrilateral circumscribed about a circle with center O, then $\angle AOB + \angle COD = 1$ st. $\angle$.

16. If a right triangle is constructed with a side of a given square as its hypotenuse, and lies outside the square, the line drawn from the vertex of the right angle to the center of the square (intersection of diagonals) bisects the right angle.

Suggestion. — Construct a circle with the hypotenuse as diameter.

17. Given $\triangle ABC$, draw MN parallel to AB and intersecting AC at M and BC at N, so that $MN = AM + BN$.

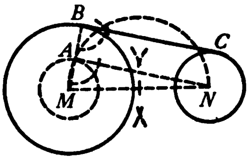
18. Given a side and an adjacent angle of a triangle, and the sum of the other two sides, construct the triangle.

Suggestion. — If AB is the given side and $\angle BAC$ the given angle, mark off AD on AC equal to the given sum of the other two sides. Draw BD. If $\triangle ABC$ is the required triangle, what is the relation between BC and CD?

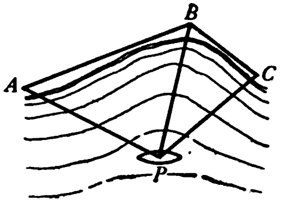
19. Construct a triangle, having given the middle points of its three sides.

20. Draw a common external tangent to two given unequal circles.

Suggestion. — If M is the center of the larger and N the center of the smaller circle, draw a circle with center M and radius equal to the difference between the radii of the circles. Draw a tangent NA to this circle from N. Draw MA and produce it to meet the given circle at B. Draw $BC \parallel AN$.

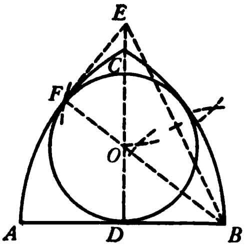


21. A ship P is in sight of three landmarks A, B, and C. The ship's officer, desiring to know the distance of the ship from the point B, observes the angles APB and BPC, and finds that $\angle APB = 82°$ and $\angle BPC = 48°$. From a map, he knows that $AB = 20$ mi. and $BC = 12$ mi. Show how by construction and measurement alone, with the map before him, he is able to find PB. If $\angle ABC = 120°$ on the map, make the construction, and find PB.



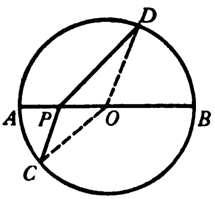
22. Find the locus of the points of contact of tangents drawn from a given point to a given set of concentric circles.

23. The Gothic arch ACB, a design frequently occurring in architecture, is constructed by drawing $\widehat{BC}$ with center A and radius AB, and drawing $\widehat{AC}$ with center B and radius AB. It is desired to inscribe a circle in this arch. Show how it may be done.



Suggestion. — Draw $ED \perp AB$ and passing through C, and make $ED = AB$. Draw EF tangent to $\widehat{AC}$. Draw FB, intersecting ED at O. Then O is the center of the required circle.

24. The shortest and longest line-segments that can be drawn to a circle from a point within the circle are the segments of the diameter through the point. Prove $PA < PC$ and $PB > PD$.

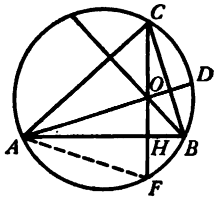


25. Construct an isosceles trapezoid, having given the altitude and the bases.

26. Find a point in a given straight line such that lines drawn from it to two given points are perpendicular to each other.

27. If the bisectors of the interior and exterior angles at a vertex of a triangle ABC meet the perpendicular bisector of the opposite side at D and E, then the five points A, B, C, D, E lie on a circle.

28. If an altitude of a triangle is produced to meet the circumscribed circle, the segment of it from the orthocenter to the circle is bisected by the side of the triangle to which it is drawn.



Suggestion. — Since OH is to be proved equal to HF, and since $OF \perp AB$, $\triangle FAO$ must be proved isosceles. For proving $\triangle FAO$ isosceles, show that $\angle BAD = \angle FAH$. This in turn requires proof that $\widehat{FB} = \widehat{DB}$. It may be proved that $\widehat{FB} = \widehat{DB}$ by first proving $\angle FCB = \angle BAD$.

29. If the altitudes AD and BE of $\triangle ABC$ intersect at M, the product of the segments into which AD is divided at M equals the product of the segments into which BE is divided.

30. If a parallel to hypotenuse AB of right triangle ABC intersects AC at D and BC at E, then $\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2$.

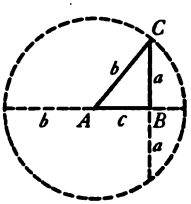
31. The diameter which bisects a chord 24 in. long is 40 in. long. Find the segments into which the diameter is divided at the point of intersection.

32. In any triangle, the difference of the squares of two sides is equal to twice the product of the third side and the projection of the median on that side.

33. The sum of the squares of the medians of a triangle is equal to three fourths of the sum of the squares of the three sides.

34. Construct the mean proportional between two given line-segments, basing the construction on the theorem in § 203.

35. Prove the *Pythagorean Theorem* by aid of the following construction: Given $\triangle ABC$ with $\angle B$ a right angle. Draw a circle with center A and radius AC. Extend CB and AB to meet the circle. Show by § 200 that $(b + c)(b - c) = a^2$. Give full proof.



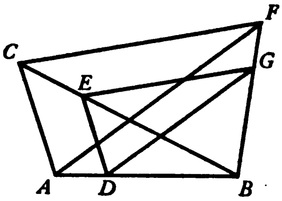
36. If through any point of the common chord of two circles two other chords are drawn, one in each circle, their four extremities all lie on a third circle.

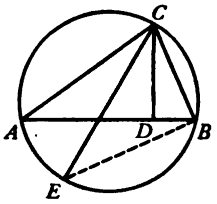
37. An isosceles trapezoid has parallel sides of lengths 12 and 16, and equal sides of length 6. Can a circle be circumscribed about this trapezoid with diameter equal to the longest side of the trapezoid?

38. If from the middle point of a common tangent of two circles a secant is drawn intersecting one circle at A and B, and another secant intersecting the other circle at C and D, then A, B, C, and D are all on one circle.

39. Prove that the square of a line drawn from the vertex of an isosceles triangle to any point in the base is equal to the square of the leg diminished by the product of the segments of the base.

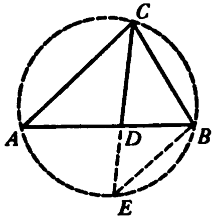
40. From D, any point in the common base AB of two triangles ABC and ABF, DE is drawn parallel to AC and DG parallel to AF, intersecting BC at E and BF at G, respectively. Prove EG parallel to CF.





41. In any triangle the product of two sides equals the product of the altitude to the third side and the diameter of the circumscribed circle.

Suggestion. — To prove $AC \times BC = CD \times CE$. Draw BE.



42. In any triangle the product of two sides equals the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

Suggestion. — To prove

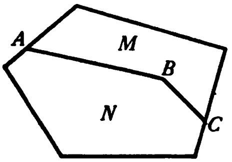
$$AC \times BC = AD \times DB + \overline{CD}^2.$$

Circumscribe a circle about $\triangle ABC$, extend CD to meet it at E, and draw EB. Prove $AC \times BC = CD \times CE = \overline{CD}^2 + CD \times DE$. Prove $CD \times DE = AD \times DB$. Use Ax. XII.

43. By aid of the formula in Ex. 42, the length of the bisector of an angle of a triangle may be computed when the three sides are known. If the sides of $\triangle ABC$ are $AB = 9$, $AC = 10$, $BC = 6$, find the bisector of $\angle C$.

Suggestion. — Find AD and DB by § 189. Substitute the values in the formula $AC \times BC = AD \times DB + CD^2$, and solve for CD.

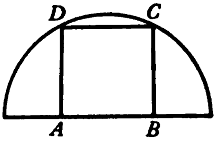
44. If AE and BF are medians of $\triangle ABC$, and intersect at D, then $\triangle BCD =$ quadrilateral $CFDE$.



45. A farm is divided into two parts, M and N, by the line ABC. Show how to replace the broken line ABC by a straight line from A, without changing the areas of M and N.

46. The opposite sides of a regular hexagon are parallel.

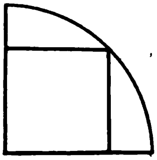
47. Construct by straightedge and compasses alone an angle of 36°; an angle of 18°.



48. The area of the square $ABCD$ inscribed in a semicircle is equal to two fifths of the area of a square inscribed in the entire circle.

49. Inscribe in a given circle a triangle similar to a given triangle.

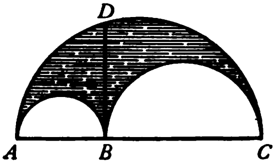
50. Circumscribe about a given circle a triangle similar to a given triangle.



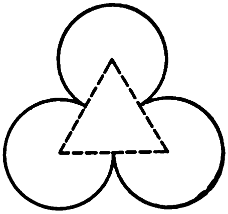
51. The area of an equilateral triangle is $18\sqrt{3}$. Find its side.

52. The area of the square inscribed in a sector of a circle whose central angle is a right angle is equal to one half of the area of a square whose side is a radius.

53. The following is an old principle, due to Archimedes: AC is divided into two unequal parts at B. Semicircles are drawn with AC, AB, and BC respectively as diameters. BD is perpendicular to AC and meets the larger semicircle at D. Then the area of the surface bounded by the three arcs AC, AB, and BC is equal to the area of the circle with diameter BD. Prove it.

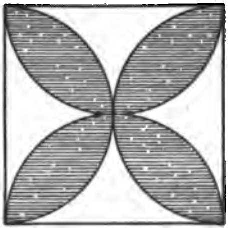


54. A *trefoil* is constructed, as in the figure, by drawing arcs of circles with centers at the vertices of an equilateral triangle and radii equal to one half of the side of the triangle. If the side of the triangle is 12 in., find the area of the trefoil, that is, of the entire surface inclosed by the three arcs. Find its perimeter.



55. If the side of the triangle in Ex. 54 is $2n$, express the area and perimeter of the trefoil.

56. The *quatrefoil* shown in the figure is formed by the arcs of four semicircles drawn with the sides of a square as diameters. Find the area and perimeter of the quatrefoil if the side of the square is 4 in.

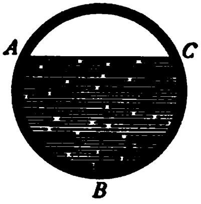


57. Find the area and perimeter of the quatrefoil in Ex. 56 if the side of the square is $2n$.

58. The perimeter of a church window is formed by three equal semicircles the centers of which are the vertices of an equilateral triangle whose sides are 4 ft. long. Find the area of the window and the length of its perimeter.

59. Consider the earth a perfect sphere, and a circular hoop made whose circumference is ten feet longer than the equator of the earth. If this hoop were placed around the equator with its center at the center of the earth, what would be the width of the ring between them? (From Yale Entrance Examination.)

60. A conduit for carrying water is 12 ft. in diameter. If the water is 9 ft. deep in the conduit, find the area of the cross section of the water. This must be known before the rate of flow of the water can be determined.



61. Find the length of the arc *ABC* in the conduit of Ex. 60. This "wetted perimeter" must be known before the resistance of the conduit to the water flow can be determined.

62. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of a regular circumscribed polygon of the same number of sides.

63. The area of a regular inscribed polygon of 12 sides equals three times the square of the radius.

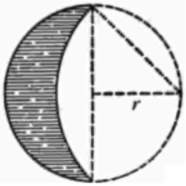
64. With each vertex of an equilateral triangle as a center and with radius equal to one half of a side, an arc is drawn within the triangle, terminating in the sides. If the side of the triangle is 6 in., find to two decimal places the area and perimeter of the surface inclosed by the three arcs.

65. On the segments of two perpendicular chords of a given circle as diameters four circles are drawn. Prove that the sum of the areas of the four circles equals the area of the given circle.

66. If r is the radius of a regular octagon, prove that the side is $r\sqrt{2-\sqrt{2}}$ and the apothem is $\frac{1}{2}r\sqrt{2+\sqrt{2}}$.

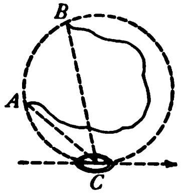
67. If r is the radius of a regular polygon of 12 sides, prove that a side is $r\sqrt{2-\sqrt{3}}$ and the apothem is $\frac{1}{2}r\sqrt{2+\sqrt{3}}$.

68. The side of a regular circumscribed octagon is less than the radius of the circle, but greater than three fourths of the radius.



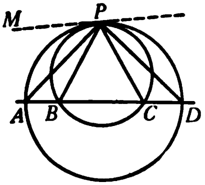
69. A crescent is bounded by a semicircle and the arc of another circle whose center is on the first arc produced. If r equals the radius of the semicircle, show that the area of the crescent is r^2.

70. Prove that a ship may be steered past a known region of danger as follows: A chart is made in which a circle is drawn through two points, A and B, which can be seen from the ship, and with sufficient radius that the circle incloses the region of danger. The inscribed angle ACB is measured. Observations of A and B from the ship are made from time to time, and the course of the ship directed so that the angle between the directions to A and B never becomes greater than $\angle ACB$.



71. If two circles are internally tangent at P, and a straight line cuts the circles in order at A, B, C, and D, prove that $\angle APB = \angle CPD$.

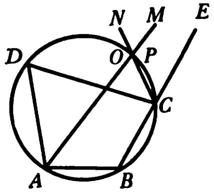
Suggestion.—$\angle MPA = \angle D$ and $\angle PCB = \angle CPD + \angle D = \angle APB + \angle MPA$.



72. If two circles intersect and parallel lines are drawn through the points of intersection, intersecting the circles, the segment intercepted on one by the two circles equals the segment intercepted on the other.

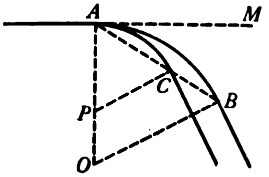
73. The straight lines which bisect an interior and an opposite exterior angle of an inscribed quadrilateral intersect on the circle.

Suggestion.—Let AM bisect $\angle BAD$ and CN bisect $\angle DCE$, and let them intersect the circle at O and P, respectively. Prove that O and P coincide.

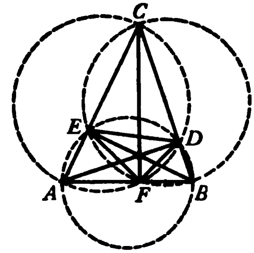


74. If the curves AC and AB of two railroad tracks begin at the same tangent point A and end in parallel tracks, prove that (1) their chords coincide in direction, and (2) their chords are proportional to their radii.

Suggestion.—Prove that $\angle BAM$ and $\angle CAM$ are equal.



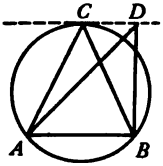
75. The line-segments joining the feet of the altitudes of a given triangle form a triangle whose angles are bisected by the altitudes of the given triangle.



Suggestions. — Draw circles with the sides as diameters. Prove that $\angle CAD = \angle EBC$, and hence that the measures of $\widehat{DC}$ and $\widehat{EC}$ are equal, etc.

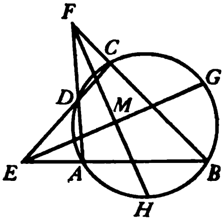
76. Construct a triangle, having given the feet of the three altitudes.

77. Of all triangles having the same base and equal altitudes, the isosceles triangle has the greatest vertical angle.



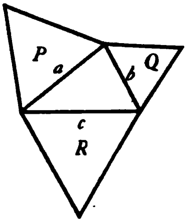
Suggestion. — Circumscribe a circle about the isosceles triangle, and apply § 182. Prove that *CD* is a tangent, and hence that *D* is outside of the circle.

78. *ABCD* is an inscribed quadrilateral. *AB* and *DC* produced meet at *E*, and *AD* and *BC* produced meet at *F*. *EG* bisects $\angle BEC$, and *FH* bisects $\angle AFB$, intersecting *EG* at *M*. Prove that $EG \perp FH$.



Suggestion. — Prove that the sum of the arcs intercepted by two vertical angles at *M* is equal to the sum of the arcs intercepted by the other two vertical angles at *M*.

79. The equilateral triangle drawn on the hypotenuse of a right triangle is equal to the sum of the equilateral triangles drawn on the two legs.



Suggestion. — $\frac{P}{R} = \frac{a^2}{c^2}$; $\frac{Q}{R} = \frac{b^2}{c^2}$; $a^2 + b^2 = c^2$.

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